
Second Edition

QUANTUM MECHANICS

Amit Goswami

Physical Constants

$$c \text{ (speed of light)} = 2.99792 \times 10^{10} \text{ cm/s}$$

$$e \text{ (|electron charge|)} = 4.803 \times 10^{-10} \text{ esu}$$

$$h \text{ (Planck's constant)} = 6.62618 \times 10^{-27} \text{ erg-s}$$

$$\hbar = h/2\pi = 1.05459 \times 10^{-27} \text{ erg-s} = 6.58217 \times 10^{-16} \text{ eV-s}$$

$$\alpha \text{ (fine structure constant)} = e^2/\hbar c = 1/137.036$$

$$m_e \text{ (electron mass)} = 9.10953 \times 10^{-28} \text{ g} = 0.511 \text{ MeV}/c^2$$

$$m_p \text{ (proton mass)} = 1.67265 \times 10^{-24} \text{ g} = 938.279 \text{ MeV}/c^2$$

$$\text{Ry (Rydberg)} = 13.60580 \text{ eV}$$

$$a_0 \text{ (Bohr radius)} = 0.52918 \times 10^{-8} \text{ cm}$$

$$\mu_B \text{ (Bohr magneton)} = e\hbar/2m_e c = 5.78838 \times 10^{-9} \text{ eV/gauss}$$

$$= 0.9273 \times 10^{-20} \text{ erg/gauss}$$

$$\mu_N \text{ (nuclear magneton)} = e\hbar/2m_p c = (1/1836)\mu_B = 5.050 \times 10^{-24} \text{ erg/gauss}$$

$$\mu_p \text{ (proton magnetic moment)} = 2.7928 \mu_N$$

$$\mu_n \text{ (neutron magnetic moment)} = -1.9103 \mu_N$$

$$N_0 \text{ (Avogadro's number)} = 6.02205 \times 10^{23}/\text{mol}$$

$$k \text{ (Boltzmann's constant)} = 1.3806 \times 10^{-16} \text{ erg/K}$$

"Those who are not shocked when they first come across quantum theory cannot possibly have understood it."
—Niels Bohr

Second Edition

QUANTUM MECHANICS

In an effort to excite college seniors and first-year graduate students about the essence of quantum mechanics, Goswami always begins a topic with what students know before moving into more complex areas. He teaches students how to ask the right questions, satisfying their interest in the meaning and interpretation of quantum mechanics, and treats the nitty-gritty details carefully. The unifying approach of the book presents quantum mechanics not only as a schema for successful calculations and predictions but also as a basis for a new and exciting worldview. The most unique aspect of the book is an ongoing presentation of the radicalness of quantum mechanics as compared to classical physics.

Outstanding features . . .

- Gives a thorough grounding in the fundamental aspects of quantum mechanics
- Provides a wealth of worked-out examples
- Offers a balanced presentation of quantum systems, helping students make a bridge to whichever aspect of modern physics they want to pursue
- Goes into enough depth without exceeding the mathematical level of the beginning student
- Liberates physics students from the bonds of classical prejudices

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To Professor Manoj K. Pal



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Preface

The intended audience for this book is college seniors and first-year graduate students. Since quantum mechanics is the most challenging course in physics for undergraduates, a textbook at this level needs to be stimulating (a touch of Feynman and Bohm), helpful in a practical way (e.g., Liboff), modern in scope (Gasiorowicz, Das and Mellissinos), elegant (Merzbacher), exhaustive and explanatory (Chester), and, most importantly, user-friendly, which, in one way or another, none of the above-mentioned books is. I hope to combine all these qualities in a book that is also user-friendly.

What makes a book user-friendly? For quantum mechanics, it means teaching how to ask the right questions, always starting a topic with what the students know, treating the nitty-gritties carefully, satisfying the students' interest in the meaning and interpretation of quantum mechanics, treating the modern topics that the students will pursue in their careers in physics, and, last but not least, giving a lot of worked-out examples.

My goals then are (1) to write a book that is stimulating like Feynman's, which excites the students about the essence of quantum mechanics; *and* (2) to give the students a thorough grounding in the fundamental aspects of quantum mechanics to help them free themselves from their classical prejudices and learn to think and calculate quantum mechanically (I include many worked-out examples and occasional summaries as practical help for the student); *and* (3) to include a lot about quantum systems in a balanced fashion that will help the student make a bridge to whichever aspect of modern physics he or she wants to pursue; *and* (4) to go to substantial depth into meaning and interpretational questions about quantum mechanics without exceeding the mathematical level of the beginning student.

To achieve these goals, I begin with the Schrödinger wave equation and its solution for one-dimensional problems, because this is mathematically more familiar to the student. Next come the Dirac bras and kets, but treated in a gentle, hand-holding way. Midway through the book spin and matrix mechanics are introduced, following the exciting style of Feynman. The book ends with three chapters on quantum systems—one chapter each on radiation and scattering theories, and one chapter on quantum measurement theory. Consequently, the book begins simply, yet excitingly. It has a second beginning (which is great for semester systems) with matrix mechanics, which recaptures the student’s interest. Finally, the book ends on a high note with both modern physics topics and interpretational questions.

The unifying approach of the book is to present quantum mechanics not only as a schema for successful calculations and predictions but also as a basis for a new and exciting world view. The most unusual aspect of the book is an ongoing presentation of the radicalness of quantum mechanics as compared to classical physics: The interference experiments, the paradoxes, nonlocality, macroscopic quantum devices, symmetry, and so forth.

I hope it is already clear that a book like this does not grow in a vacuum. I have taken help from all the books that I myself have found worthwhile (including but not limited to those already mentioned); the books especially useful in writing a particular chapter have been referenced at the end of that chapter. I am also grateful to my students Mark Mitchell and John Svitak, and to Arthur Pavlovic of West Virginia University, Philip N. Parks of Michigan Technological University, Sanford Kern of Colorado State University, and Walter Carnahan of Indiana State University, the reviewers who toiled to make the book error-free. I also thank my editor, Jeffrey L. Hahn, for his encouragement. Finally, heartfelt thanks are due my wife, Maggie, for a careful editing of the book.

.....

PREFACE TO THE SECOND EDITION

I am grateful to the many users of this book who I credit for creating the need for a second edition. You like the user-friendliness; you like the emphasis on meaning. Thank you.

The temptation to add more material was a serious one that I actively resisted because most of you are very happy with the current size of the book. *So the only substantial change I offer is the addition of many more problems at the end of the chapters.* Also, of course, errors have been corrected (I hope, every one of them!). My thanks for the generous feedback on these errors. I have also added clarification wherever needed. I hope these changes make the book even more user-friendly.

Finally, I’d like to thank my colleague, Professor Robert Zimmerman, for a careful reading of the revised manuscript, the staff for their patience, and Jan Blankenship and my wife Maggie for their help.

1

An Introduction to the Schrödinger Equation

What is quantum mechanics? It's a new way of interpreting data and predicting the behavior of submicroscopic objects, perhaps of all objects, based on the idea of an essential discontinuity, the quantum, in the affairs of the world. Compared to the world view of classical physics, quantum mechanics gives us a radically new view of the world. Indeed, it will revolutionize the way you think about the world.

The discovery of quantum mechanics proceeded along two important but separate tracks. The first track was based on the realization that the allowed values of energy exchange between subatomic bodies are discrete—they involve quantum jumps. This track began with Max Planck's work on black-body radiation, got a big boost from Niels Bohr's theory of the hydrogen atom, and was carried to fruition by Werner Heisenberg's discovery of the matrix version of quantum mechanics.

The second track began with Albert Einstein's discovery of the wave-particle duality of light. Then Louis de Broglie generalized the attribute of wave-particle duality to matter, Erwin Schrödinger discovered the wave equation for matter waves, now called the Schrödinger equation, and Max Born interpreted the de Broglie-Schrödinger waves as waves of probability. This was the wave-mechanics version of quantum mechanics. Finally, Paul Dirac showed that the two versions, matrix mechanics and wave mechanics, are entirely equivalent.

However different the two tracks may look, there is one thing they have in common—they troubled many people, they continue to shock people, they should shock you! The Planck-Bohr-Heisenberg discoveries introduced quantum jumps, discontinuities, in physics. The Einstein-de Broglie-Schrödinger-

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