PHILOSOPHY OF MATHEMATICS

A Contemporary Introduction to the World of Proofs and Pictures

Second Edition

James Robert Brown

Routledge
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PHILOSOPHY OF MATHEMATICS

In his long-awaited new edition of *Philosophy of Mathematics*, James Robert Brown tackles important new as well as enduring questions in the mathematical sciences. Can pictures go beyond being merely suggestive and actually prove anything? Are mathematical results certain? Are experiments of any real value?

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For Elizabeth and Stephen

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Preface and Acknowledgements

> A philosopher who has nothing to do with geometry is only half a philosopher, and a mathematician with no element of philosophy in him is only half a mathematician. These disciplines have estranged themselves from one another to the detriment of both.
>
> Frege

> A heavy warning used to be given that pictures are not rigorous; this has never had its bluff called and has permanently frightened its victims into playing for safety. Some pictures, of course, are not rigorous, but I should say most are (and I use them whenever possible myself).
>
> Littlewood

There are a number of ways in which this book could fail. It has several goals, some of them pedagogical. One of these goals is to introduce readers to the philosophy of mathematics. In my attempt to avoid failure here I've included chapters on traditional points of view, such as formalism and constructivism, as well as Platonism. And since I'm aiming at a broad audience, I've taken pains to explain philosophical notions that many readers may encounter for the first time. I've also given lots of detailed mathematical examples for the sake of those who lack a technical background. It's been my experience that there is a huge number of students who come to philosophy from a humanities background wanting to know a bit about the sciences, and when they are properly introduced they find that their appetites for mathematics become insatiable. I'd be delighted to stimulate a few readers in this way.

If we taught philosophy today in a way that reflected its history, the current curriculum would be overwhelmed with the philosophy of mathematics. Think of these great philosophers and how important mathematics is to their thought: Plato, Descartes, Leibniz, Kant, Frege, Russell, Wittgenstein, Quine, Putnam, and so many others. And interest in the nature of mathematics is not confined to the so-called analytic stream of philosophy; it also looms large in the work of Husserl and Lonergin, central figures in, respectively, the continental and Thomistic philosophical traditions. Anyone sincerely interested in philosophy must be interested in the nature of mathematics, and I hope to show why. As for those who persist in thinking otherwise – let them burn in hell.

This book could also fail in a second, more important aim, which is to introduce some of the newer issues in the philosophy of mathematics, namely those associated with computers, 'experimentation', and especially with visualization. Traditional issues remain fascinating and unresolved; philosophers and mathematicians alike continue to work on them. (Even logicism, the view that mathematics is really just logic, is making a partial comeback.) But if there are living philosophical issues for working mathematicians, they have to do with the role of computers and computer graphics and the role of physics within mathematics. Some consider the use of computers a glorious revolution – others think it a fraud. Some are thrilled at the new relations with physics – others fear the fate of rigour. Current battles are just as lively as those between Russell and Poincaré early in the twentieth century or between Hilbert and Brouwer in the 1920s and 1930s. And philosophers should know about them. This book would be a failure if something of the content of the issues and the spirit of current debates is not conveyed.

Finally, I could fail in my attempt to argue for Platonism, in general, and, for a Platonistic account of how (some) pictures work, in particular. Mathematicians are instinctively realists; but when forced to think about the details of this realism, they often become uneasy. Philosophers, aware of the bizarreness of abstract objects, are already wary of mathematical realism. But still, most people are somewhat sympathetic to Platonism in mathematics, tolerant to an extent that they wouldn't tolerate, say, Platonism in physics or in ethics. My case for Platonism will meet with at least mild resistance, but this is nothing compared with the hostility that will greet my account of how picture-proofs work. On this last point I expect to fail completely in winning over readers. But I will be somewhat mollified if it is generally admitted that the problems this work raises and addresses are truly wonderful, worthy of wide attention.

I've had a great deal of help from a great many people in a great many ways. Some are long-time colleagues with whom I've been arguing these issues longer than we care to remember. Some are students subjected to earlier drafts. Some listened to an argument. Some read a chapter. Some worked carefully through the whole of an earlier draft. For their help in whatever form, enormous thanks go to: Peter Apostoli, Michael Ashooh, John Bell, Gordon Belot, Alexander Bird, Elizabeth East, Danny Goldstick, Ian Hacking, Michael Hallett, Sarah Hoffman, Andrew Irvine, Loki Jorgenson, Bernard Katz, Margery Konan, Hugh Lehman, Mary Leng, Dennis Lomas, Ken Manders, James McAllister, Patrick

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Chapters 3, 4 and 7 are revised from earlier articles. I thank Oxford University Press and Kluwer Academic Publishers for their kind permission to use this material.

Finally, I'm very grateful to SSHRC for its support.

Preface to the Second Edition

This edition differs from the first in several respects. I have made numerous minor modifications and corrections throughout. I'm very grateful to all those who pointed out mistakes or unclear passages. In a few places I have overhauled whole paragraphs. And I have added a brief 'Further Reading' section to the end of each chapter.

The biggest change comes in the form a new chapter on the continuum hypothesis (Chapter 11). This has been one of the great problems of mathematics for more than a century. It was shown by Gödel and Cohen to be independent, hence neither provable nor refutable, given the other axioms of set theory. Realists claim it has a truth-value, nevertheless. Could we ever come to know what that truth value is? Christopher Freiling may have refuted it by means of a thought experiment. This is certainly not the usual way of doing mathematics, but if it works – and I'm inclined to think it does – then it wonderfully illustrates the power of new techniques, such as visual reasoning.

I expect this new chapter to be controversial – indeed, I hope it is. Even if it fails as a refutation of the continuum hypothesis, I'll be gratified if it provokes deeper reflection on the cluster of issues associated with the continuum hypothesis and with visual thinking in mathematics, in general. That remains a main aim throughout the whole book.

I'm very pleased with the reception of the first edition. The reviewers were overly generous (a fault I'm happy to pardon). Students seemed to get something out of it and even enjoyed doing so. Experts found things to contest. I couldn't reasonably ask for more – but I will. If I had any disappointment, it would concern the more esoteric topics. For instance, reviewers often remarked enthusiastically on the potential interest and importance of topics such as notation. But as far as I know, the theme has not been further explored. I suppose all I can do is once again urge others to take up the matter. The old topics such as constructivism and formalism remain interesting, and so are the newer ones such as indispensability and structuralism. But there is a goldmine waiting for us all in the issues of visualization, notation, computer simulation, and mathematical thought experimentation. That's where the future lies.

A number of people need to be thanked. I reiterate my thanks to those who helped with the first edition. Some of these and some others were very helpful this time, as well. In particular, thanks go to Ken Manders and Louis Levin for finding mistakes and typos, to Zvonimir Šikić for critical comments on graphs, and Chris Freiling for comments on my account of his work on the continuum. I also learned a great deal from reviewers of the first edition. A list of these can be found at http://www.chass.utoronto.ca/~jrbrown/NOTES.Philosophy_of_Mathematics. Any additional comments, corrections, and reviews of this second edition will also be listed there.

CHAPTER 1
Introduction: The Mathematical Image

Let's begin with a nice example, the proof that there are infinitely many prime numbers. If asked for a typical bit of real mathematics, your friendly neighbourhood mathematician is as likely to give this example as any. First, we need to know that some numbers, called 'composite', can be divided without remainder or broken into factors (e.g. $6 = 2 \times 3$, $561 = 3 \times 11 \times 17$), while other numbers, called 'prime', cannot (e.g. 2, 3, 5, 7, 11, 13, 17, . . .). Now we can ask: How many primes are there? The answer is at least as old as Euclid and is contained in the following.

> *Theorem*: There are infinitely many prime numbers.
>
> *Proof*: Suppose, contrary to the theorem, that there is only a finite number of primes. Thus, there will be a largest which we can call p. Now define a number n as 1 plus the product of all the primes:
>
> $$n = (2 \times 3 \times 5 \times 7 \times 11 \times \ldots \times p) + 1$$
>
> Is n itself prime or composite? If it is prime then our original supposition is false, since n is larger than the supposed largest prime p. So now let's consider it composite. This means that it must be divisible (without remainder) by prime numbers. However, none of the primes up to p will divide n (since we would always have remainder 1), so any number which does divide n must be greater than p. This means that there is a prime number greater than p after all. Thus, whether n is prime or composite, our supposition that there is a largest prime number is false. Therefore, the set of prime numbers is infinite.

The proof is elegant and the result profound. Still, it is typical mathematics; so, it's a good example to reflect upon. In doing so, we will begin to see the elements of *the mathematical image*, the standard conception of what mathematics is. Let's begin a list of some commonly accepted aspects. By 'commonly accepted' I mean that they would be accepted by most working mathematicians, by most educated people, and probably by most philosophers of mathematics, as well. In listing them as part of the common mathematical image we need not endorse them. Later we may even come to reject some of them – I certainly will. With this caution in mind, let's begin to outline the standard conception of mathematics.

Certainty The theorem proving the infinitude of primes seems established beyond a doubt. The natural sciences can't give us anything like this. In spite of its wonderful accomplishments, Newtonian physics has been overturned in favour of quantum mechanics and relativity. And no one today would bet too heavily on the longevity of current theories. Mathematics, by contrast, seems the one and only place where we humans can be absolutely sure we got it right.

Objectivity Whoever first thought of this theorem and its proof made a great discovery. There are other things we might be certain of, but they aren't discoveries: 'Bishops move diagonally.' This is a chess rule; it wasn't discovered; it was invented. It is certain, but its certainty stems from our resolution to play the game of chess that way. Another way of describing the situation is by saying that our theorem is an objective truth, not a convention. Yet a third way of making the same point is by saying that Martian mathematics is like ours, while their games might be quite different.

Proof is essential With a proof, the result is certain; without it, belief should be suspended. That might be putting it a bit too strongly. Sometimes mathematicians believe mathematical propositions even though they lack a proof. Perhaps we should say that without a proof a mathematical proposition is not justified and should not be used to derive other mathematical propositions. Goldbach's conjecture is an example. It says that every even number is the sum of two primes. And there is lots of evidence for it, e.g. $4 = 2 + 2$, $6 = 3 + 3$, $8 = 3 + 5$, $10 = 5 + 5$, $12 = 7 + 5$, and so on. It's been checked into the billions without a counter-example. Biologists don't hesitate to conclude that all ravens are black based on this sort of evidence; but mathematicians (while they might believe that Goldbach's conjecture is true) won't call it a theorem and won't use it to establish other theorems – not without a proof.

Let's look at a second example, another classic, the Pythagorean theorem. The proof below is modern, not Euclid's.

> *Theorem*: In any right-angled triangle, the square of the hypotenuse is equal to the sum of the squares on the other two sides.

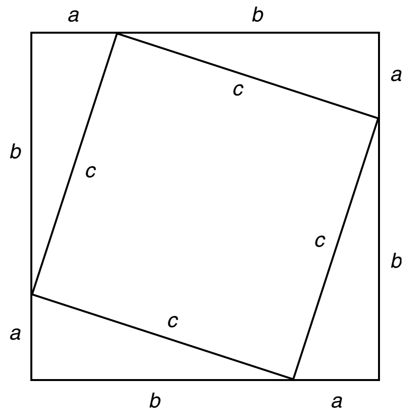

Figure 1.1

> *Proof*: Consider two square figures, the smaller placed in the larger, making four copies of a right-angled triangle $\triangle abc$ (Figure 1.1). We want to prove that $c^2 = a^2 + b^2$.
>
> The area of the outer square $= (a + b)^2 = c^2 + 4 \times$ (area of $\triangle abc$) $= c^2 + 2ab$, since the area of each copy of $\triangle abc$ is $\frac{1}{2}ab$. From algebra we have $(a + b)^2 = a^2 + 2ab + b^2$. Subtracting $2ab$ from each, we conclude $c^2 = a^2 + b^2$.

This brings out another feature of the received view of mathematics.

Diagrams There are no illustrations or pictures in the proofs of most theorems. In some there are, but these are merely a psychological aide. The diagram helps us to understand the theorem and to follow the proof – nothing more. The proof of the Pythagorean theorem or any other is the verbal/symbolic argument. Pictures can never play the role of a real proof.

Remember, in saying this I'm not endorsing these elements of the mathematical image, but merely exhibiting them. Some of these I think right, others, including this one about pictures, quite wrong. Readers might like to form their own tentative opinions as we look at these examples.

Misleading diagrams Pictures, at best, are mere psychological aids; at worst they mislead us – often badly. Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$$

which we can illustrate with a picture (Figure 1.2):

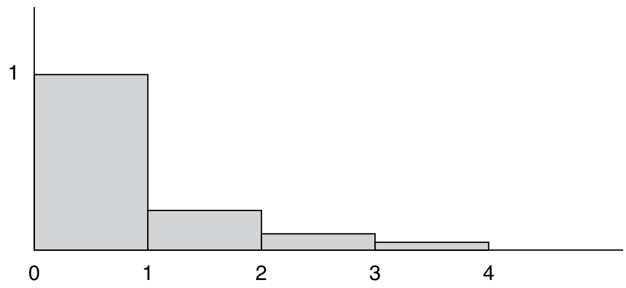

Figure 1.2 Shaded blocks correspond to terms in the series

The sum of this series is $\pi^2/6 = 1.6449\ldots$ In the picture, the sum is equal to the shaded area. Let's suppose we paint the area and that this takes *one* can of paint.

Next consider the so-called harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$$

Here's the corresponding picture (Figure 1.3):

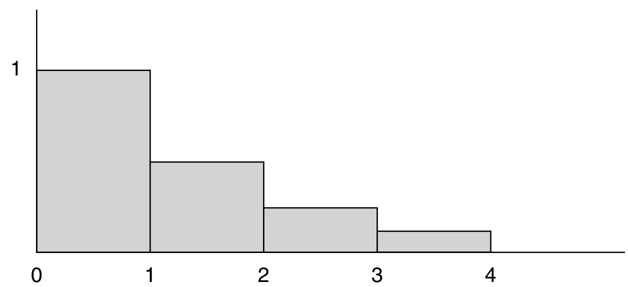

Figure 1.3

The steps keep getting smaller and smaller, just as in the earlier case, though not quite so fast. How big is the shaded area? Or rather, how much paint will be required to cover the shaded area? Comparing the two pictures, one would be tempted to say that it should require only slightly more – perhaps two or three cans of paint at most. Alas, such a guess couldn't be further off the mark. In fact, there isn't enough paint in the entire universe to cover the shaded area – it's infinite. The proof goes as follows. As we write out the series, we can group the terms:

$$\underbrace{\frac{1}{1}} + \underbrace{\frac{1}{2} + \frac{1}{3}} + \underbrace{\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}} + \underbrace{\frac{1}{8} + \frac{1}{9} + \ldots}$$

The size of the first group is obviously 1. In the second group the terms are between $\frac{1}{2}$ and $\frac{1}{4}$, so the size is between $2 \times \frac{1}{2}$ and $2 \times \frac{1}{4}$, that is, between $\frac{1}{2}$ and 1. In the next grouping of four, all terms are bigger than 1/8, so the sum is again between $\frac{1}{2}$ and 1. The same holds for the next group of 8 terms; it, too, has a sum between $\frac{1}{2}$ and 1. Clearly, there are infinitely many such groupings, each with a sum greater than $\frac{1}{2}$. When we add them all together, the total size is infinite. It would take more paint than the universe contains to cover it all. Yet, the picture doesn't give us an inkling of this startling result.

One of the most famous results of antiquity still amazes; it is the proof of the irrationality of the square root of 2. A rational number is a ratio, a fraction, such as 3/4 or 6937/528, which is composed of whole numbers. $\sqrt{9} = 3$ is rational and so is $\sqrt{(9/16)} = 3/4$; but $\sqrt{2}$ is not rational as the following theorem shows.

Theorem: The square root of 2 is not a rational number.

Proof: Suppose, contrary to the theorem, that $\sqrt{2}$ is rational, i.e. suppose that there are integers p and q such that $\sqrt{2} = p/q$. Or equivalently, $2 = (p/q)^2 = p^2/q^2$. Let us further assume that p/q is in lowest terms. (Note that $3/4 = 9/12 = 21/28$, but only the first expression is in lowest terms.)

Rearranging the above expression, we have $p^2 = 2q^2$. Thus, p^2 is even (because 2 is a factor of the right side). Hence, p is even (since the square of an odd number is odd). So it follows that $p = 2r$, for some number r. From this we get $2q^2 = p^2 = (2r)^2 = 4r^2$. Thus, $q^2 = 2r^2$, which implies that q^2 is even, and hence that q is even.

Now we have the result that both p and q are even, hence both divisible by 2, and so, not in lowest terms as was earlier supposed. Thus, we have arrived at the absurdity that p/q both *is* and *is not* in lowest terms. Therefore, our initial assumption that $\sqrt{2}$ is a rational number is false.

Classical logic Notice the structure of the proof of the irrationality of $\sqrt{2}$. We made a supposition. We derived a contradiction from this, showing the supposition is false. Then we concluded that the negation of the supposition is true. The logical principles behind this are: first, no proposition is both true and false (non-contradiction) and second, if a proposition is false, then its negation is true (excluded middle). Classical logic is a working tool of mathematics. Without this tool, much of traditional mathematics would crumble.

Strictly speaking, the proof of the irrationality of $\sqrt{2}$ is acceptable to constructive mathematicians, even though they deny the general legitimacy of classical logic. The issue will come up in more detail in a later chapter. The proof just given nicely illustrates *reduction ad absurdum* reasoning. It is also one of the all time great results, which everyone should know as a matter of general culture, just as everyone should know *Hamlet*. This is my excuse for using an imperfect example to make the point about classical logic.

Sense experience All measurement in the physical world works perfectly well with rational numbers. Letting the standard metre stick be our unit, we can measure any length with whatever desired accuracy our technical abilities will allow; but the accuracy will always be to some rational number (some fraction of a metre). In other words, we could not discover irrational numbers or incommensurable segments (i.e. lengths which are not ratios of integers) by physical measurement. It is sometimes said that we learn $2 + 2 = 4$ by counting apples and the like. Perhaps experience plays a role in grasping the elements of the natural numbers. But the discovery of the irrationality of $\sqrt{2}$ was an intellectual achievement, not at all connected to sense experience.

Cumulative history The natural sciences have revolutions. Cherished beliefs get tossed out. But a mathematical result, once proven, lasts forever. There are mathematical revolutions in the sense of spectacular results which yield new methods to work with and which focus attention in a new field – but no theorem is ever overturned. The mathematical examples I have so far discussed all predate Ptolemaic astronomy, Newtonian mechanics, Christianity and capitalism; and no doubt they will outlive them all. They are permanent additions to humanity's collection of glorious accomplishments.

Computer proofs Computers have recently played a dramatic role in mathematics. One of the most celebrated results has to do with map colouring. How many colours are needed to insure that no adjacent countries are the same colour?

> *Theorem*: Every map is four-colourable.

I won't even try to sketch the proof of this theorem. Suffice it to say that a computer was set the task of checking a very large number of cases. After a great many hours of work, it concluded that there are no counter-examples to the theorem: every map can be coloured with four colours. Thus, the theorem was established.

It's commonplace to use a hand calculator to do grades or determine our finances. We could do any of these by hand. The little gadget is a big time saver and often vastly more accurate than our efforts. Otherwise, there's really nothing new going on. Similarly, when a supercomputer tackles a big problem and spends hours on its solution, there is nothing new going on there either. Computers do what we do, only faster and perhaps more accurately. Mathematics hasn't changed because of the introduction of computers. A proof is still a proof, and that's the one and only thing that matters.

Solving problems There are lots of things we might ask, but have little chance of answering: 'Does God exist?' 'Who makes the best pizza?' These seem perfectly meaningful questions, but the chances of finding answers seems hopeless.

By contrast, it seems that every mathematical question can be answered and every problem solved. Is every even number (greater than 2) equal to the sum of two primes? We don't know now, but that's because we've been too stupid so far. Yet we are not condemned to ignorance about Goldbach's conjecture the way we are about the home of the best pizza. It's the sort of question that we should be able to answer, and in the long run we will.

Having said this, a major qualification is in order. In fact, we may have to withdraw the claim. So far, in listing the elements of the mathematical image we've made no distinction among mathematicians, philosophers and the general public. But at this point we may need to distinguish. Recent results such as Gödel's incompleteness theorem, the independence of the continuum hypothesis and others have led many mathematicians and philosophers of mathematics to believe that there are problems which are unsolvable in principle. The pessimistic principle would seem to be part of the mathematical image.

Well, enough of this. We've looked at several notions that are very widely shared and, whether we endorse them or not, they seem part of the common conception of mathematics. In sum, these are a few of the ingredients in the mathematical image:

(1) *Mathematical results are certain*
(2) *Mathematics is objective*
(3) *Proofs are essential*
(4) *Diagrams are psychologically useful, but prove nothing*
(5) *Diagrams can even be misleading*
(6) *Mathematics is wedded to classical logic*
(7) *Mathematics is independent of sense experience*
(8) *The history of mathematics is cumulative*
(9) *Computer proofs are merely long and complicated regular proofs*
(10) *Some mathematical problems are unsolvable in principle*

More could be added, but this is grist enough for our mill. Here we have the standard conception of mathematics shared by most mathematicians and non-mathematicians, including most philosophers. Yet not everyone accepts this picture. Each of these points has its several critics. Some deny that mathematics was ever certain and others say that, given the modern computer, we ought to abandon the ideal of certainty in favour of much more experimental mathematics. Some deny the objectivity of mathematics, claiming that it is a human invention after all, adding that though it's a game like chess, it is the greatest game ever played. Some deny that classical logic is indeed the right tool for mathematical inference, claiming that there are indeterminate (neither true nor false) mathematical propositions. And, finally, some would claim great virtues for pictures as proofs, far beyond their present lowly status.

We'll look at a number of issues in the philosophy of mathematics, some traditional, some current, and we'll see how much of the mathematical image endures this scrutiny. Don't be surprised should you come to abandon at least some of it. I will.

Further Reading

Many come to the philosophy of mathematics before a serious encounter with mathematics itself. If you're looking for a good place to get your feet wet, try an old classic, by Courant, Robins, and Stewart, *What is Mathematics*? If you're trying to teach yourself mathematics using standard textbooks, then I strongly urge reading popular books, as well. Rough analogies, anecdotes, and even gossip are an important part of any mathematical education. Biographies are important, too. For a collection of brief biographies of several contemporaries, try Albers and Alexanderson (eds) *Mathematical People*. There are several introductory books in the philosophy of mathematics. Shapiro, *Talking About Mathematics* is particularly nice; it covers traditional topics and Shapiro's own 'structuralism'.

CHAPTER 2
Platonism

What's the greatest discovery in the history of thought? Of course, it's a silly question – but it won't stop me from suggesting an answer. It's Plato's discovery of abstract objects. Most scientists, and indeed most philosophers, would scoff at this. Philosophers admire Plato as one of the greats, but think of his doctrine of the heavenly forms as belonging in a museum. Mathematicians, on the other hand, are at least slightly sympathetic. Working day-in and day-out with primes, polynomials and principal fibre bundles, they have come to think of these entities as having a life of their own. Could this be only a visceral reaction to an illusion? Perhaps, but I doubt it. The case for Platonism, however, needs to made carefully. Let's begin with a glance at the past.

The Original Platonist

We notice a similarity among various apples and casually say, 'There is something they have in common.' But what could this *something* they have in common be? Should we even take such a question literally? Plato did and said the common thing is *the form of an apple*. The form is a perfect apple, or perhaps a kind of blueprint. The actual apples we encounter are copies of the form; some are better copies than others. A dog is a dog in so far as it 'participates' in *the form of a dog*, and an action is morally just in so far as it participates in *the form of justice*.

How do we know about the forms? Our immortal souls once resided in heaven and in this earlier life gazed directly upon the forms. But being born into this world was hard on our memories; we forgot everything. Thus, according to Plato, what we call learning is actually recollection. And so, the proper way to teach is the so-called Socratic method of questioning, which