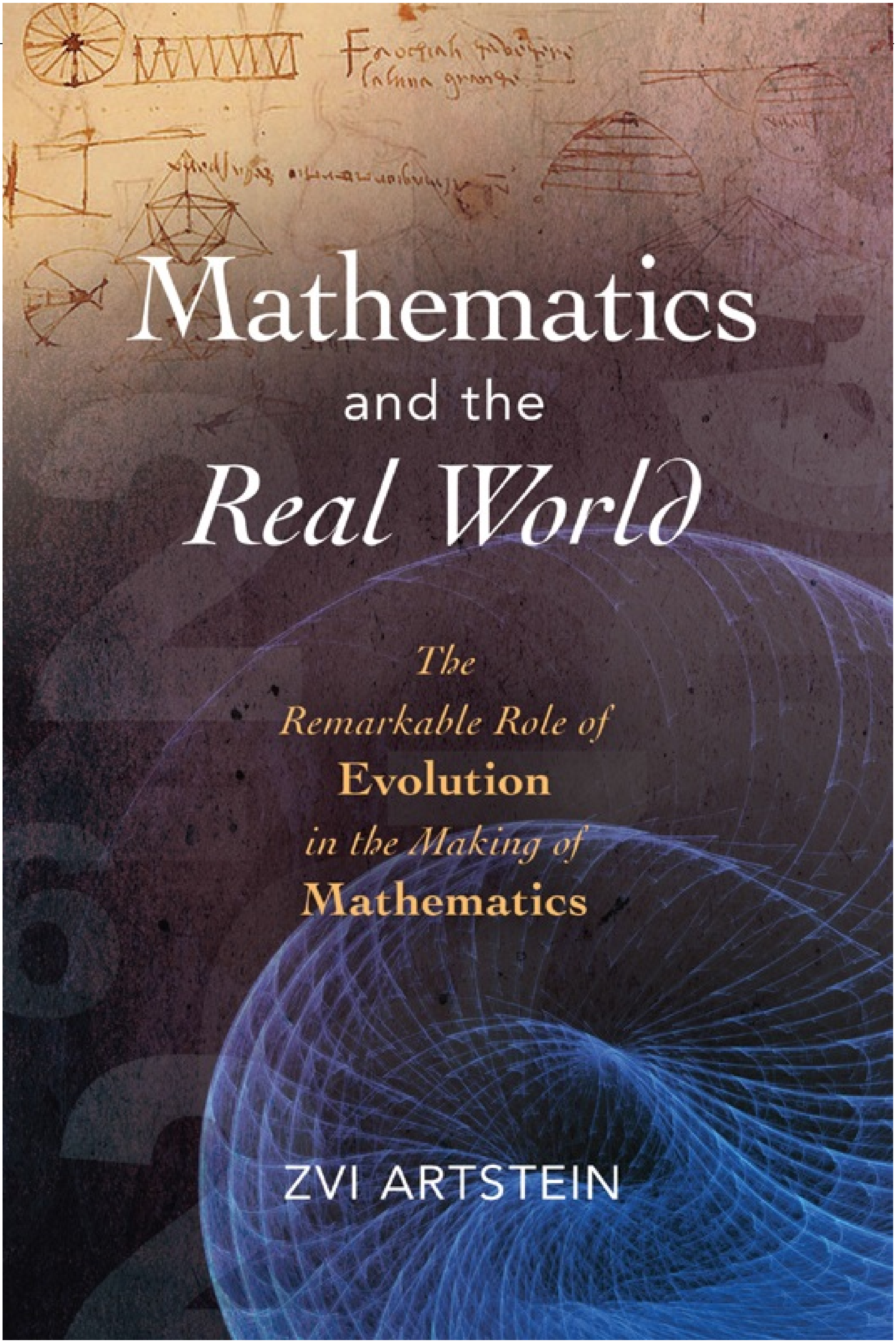
Mathematics
and the
Real World
The
Remarkable Role of
Evolution
in the Making of
Mathematics
ZVI ARTSTEIN

MATHEMATICS AND THE REAL WORLD

THE REMARKABLE ROLE OF EVOLUTION IN THE MAKING OF MATHEMATICS

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TRANSLATED FROM HEBREW BY ALAN HERCBERG

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To Yael

CONTENTS

PREFACE

There are many jokes about mathematicians. One of my favorites is about an engineer, an architec and a mathematician who have been sentenced to be hanged. In the evening before the day set for th execution, the warden asks them for their last requests. The engineer asks to be allowed to present new machine he has designed that can perform all household chores without any human interventio The warden promises that the next day, before the hanging, he will have one hour in which to show h machine to the prison staff and his two fellow death-row inmates. The architect asks to be allowed explain his new concept of residential accommodation, a modern house that keeps cool in the summ and warm in the winter, without expenditure on fuel. Again the warden promises that the next da before the hanging, he will have one hour in which to present his idea to the prison staff and his tw fellow death-row prisoners. The mathematician says that he has recently proved a mathematic theorem that will shake the foundations of mathematics, and he would like to reveal it in a lecture an intelligent audience. The warden starts agreeing…and the engineer and the architect start shoutin “We want our execution to be brought forward to this evening!”

This joke appeals to me because it reflects the widespread attitude of the public to what can b expected from books and lectures on mathematics. We will give the reasons for this attitude later, ar here we will just note that even in school we are exposed to indoctrination that causes us to rela differently to texts and lectures on mathematics than we do to other subjects. In school students a expected to solve mathematical exercises to show that they have understood the material. Oth subjects such as history, literature, or even biology do not require such exercises. The impression th this creates is that without solving exercises there is no point in listening to mathematics. Th development of intuitive understanding of a subject, without practicing what has been learned, is n accepted as understanding in the case of mathematics. That is so despite the fact that an intuitive gras of a subject, without needing to put it into practice, is an acceptable objective in other scientific ar general disciplines. This is misguided and misleading indoctrination that does an injustice mathematics. Furthermore, that approach is alien to professional mathematicians too. Of course the must have a deep understanding of the topics they are researching, but an intuitive understanding other mathematical subjects is sufficient. I will put forward an analogy that I would ask you to keep mind as you read this book.

I love classical music and regularly attend concerts of the Israel Philharmonic Orchestra, and greatly enjoy both live performances and recordings. I cannot read music, and I do not know th detailed history of music or the life stories of the different composers. I am confident that those wh can read music or are familiar with the history of music enjoy what they hear in a way that is differe from my enjoyment. I am not sure if they enjoy it more than I do because, for example, they may b conscious of any note played slightly inaccurately, whereas I would be totally oblivious of it. Th experts understand the compositions on different levels from mine, but I enjoy the music immensel perhaps not from the written notes, but from the tune. Not the trees, but the forest. There are hard any “notes” in this book, nor trees, mainly a tune, mainly the forest. If one or a few notes appear he or there (at times using a different font, and preceded by a rule line), they can be skipped witho breaking the thread of the text.

The different sections of the book are connected, but the ideas are presented in such a way that eac section is self-contained and can be read independently of the others. The headings and titles of th sections and chapters indicate the central elements within them. It is advisable to start with chapter but then the reader can certainly go straight to the chapter on the mathematics of randomness or to th one on the mathematics of human behavior, or even jump to the last chapter on teaching mathematics

Naturally, a book like this could not have been written without information, exchange of views, ar help ~~that I received from friends, colleagues, students, those who heard lectures on the topics covere~~ in the book that I delivered in various forums, the translator and the editor, the publisher's team, an of course, my family. There are too many people for me to be able to enumerate each one of the here. To all of them, my sincere thanks.

So what is it all about: The books deals with *the mathematics of nature*, *the nature of mathematics*, ar their interrelationship. We will describe, by means of a historical review as well as from the aspect c current research, the link between mathematics and the physical world and the social world around u The discussion in the book also relates to areas of science and society to which mathematics relevant. We will therefore also present scientific facts and social situations described b mathematics. That presentation is not exhaustive or detailed, as we focus on the mathematical aspec of the various fields. The discussion will be accompanied by the constant presence of the questic regarding the extent of the effect of the *evolution of the human race* on the development mathematics and its applications. We will examine the claim that the manner in which the huma brain was fashioned by millions of years of evolution affected humans' mathematical capabilities ar the type of mathematics that is easy for humans to develop and understand. We will also show that, t a large extent, evolution is responsible for the difficulty we have in understanding certain other are of mathematics. We will try to do all that with a minimum of musical notes but with much pleasir music.

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CHAPTER I

EVOLUTION, MATHEMATICS, AND THE EVOLUTION OF MATHEMATICS

Could evolution have affected mathematics? • Can horses perform calculations? • Can rats count? • Do infants solve problems of addition and subtraction? • Which rectangles please us? • Why are clowns scary? • What color are the sheep in Ireland? • What number comes next in the sequence 4, 14, 23, 34, 42, 50, 59,…? • Why square the circle? • How have optical illusions contributed to science?

1. EVOLUTION

The theory of evolution is attributed to Charles Darwin, but it was not Darwin who initiated the stud of evolution. King Solomon stated, “There is nothing new under the sun,” (Ecclesiastes 1:9), philosophical statement alluding to the observation that the world is in a constant state of flux. At an given time we see the current situation around us, but we also follow changes that take place in o lifetime, and we are aware of changes occurring over periods of time that we are unable to obser directly. The evidence regarding changes that took place in the past often enables us to infer wh caused those changes. That applies both to the physical world, such as rocks, flora, and fauna, and society, including modes of behavior, fashion, literature, medical practices, and technology. Th changes take place according to their own mechanism. Sometimes it is clear to us what survives, wh is modified, and what becomes extinct, but it is not always easy to identify the mechanism.

Let us take as an example the Earth's surface. Some rocks exist for many years, while others a weathered and eroded by the wind almost as we watch. What causes the difference? Clearly it is t different rock textures that determine the differences in their ability to survive. Basalt will last, whi limestone will crumble. There are no sand dunes on the tops of mountains because they would b blown away by the wind. We could say that the strong triumph, the fittest survive. We can deduce tha being made of basalt is an advantage in the battle for survival on a mountain peak. That statement trivial in the realm of rocks, and we do not usually examine rocks in terms of the competition f survival. But the conclusion that whatever is most suited to its environment survives is corre regarding rocks as well as human society. Historians, in discussing human history, try to understar why a particular society survived and another disappeared. Their conclusions generally refer to t advantages that the victors had over the vanquished. We can learn about the conditions under which society or species developed from its specific characteristics. Likewise, from the conditions in whic it developed, we can learn about the advantages that enabled it to win the battle for survival.

Darwin's great contribution to the theory of evolution was in identifying the mechanism by whic different species of animals and plants changed and developed. Unlike Lamarck, who claimed th every species adapts to its environment and the characteristics it takes on are passed from eac generation to the next, Darwin proposed a different mechanism responsible for the changes that eve species undergoes. The mechanism consists of two main elements: mutation and selection. In t reproduction process, individuals undergo mutations that cause random and generally minor chang in their characteristics. The individuals with the best-suited characteristics reproduce at the faste rate, and that constitutes the selection that results in successive generations of each species bein better suited to the environmental conditions. The best-suited species among those competing for t same food resources are the ones that survive.

Charles Robert Darwin (1809–1882) was born in the town of Shrewsbury, England, to a wel established family. His father, Robert, was a wealthy physician. His grandfather, Erasmus Darwi who died before Charles was born but whose writings were available to Charles, was a philosopher ar a naturalist who favored the theory of evolution of the leading French naturalist Jean-Baptiste d Monet, Chevalier de Lamarck (1744–1829), known simply as Lamarck. Young Charles was exposed t scientific endeavors but was not a particularly industrious student. Instead of devoting his time to h studies, he preferred to observe nature and to collect various items, particularly beetles of differe types. When he was twenty-three he was invited to join an expedition due to sail on a ship called th *Beagle* as its scientist; the main purpose of the expedition was to chart the shores of Australia ar South America for the British Empire. His role was to collect and classify geological, zoological, ar botanical specimens. In the course of the voyage Darwin noted that different but similar species cou be found living in regions near each other, specifically on the Galapagos Islands. It was there that h conceived the model of evolution consisting of mutations and selection. It should be noted th Darwin, in developing his theory of the evolution of types of flora and fauna, was deeply influence by the theory of the political philosopher Thomas Malthus (who lived half a century before him) o the demographic and economic development of human societies. Darwin's autobiography shows hi to have been a modest and wise man; it is replete with statements indicating a deep understanding evolution beyond the technique he developed. For example, in his reference to his elderly colleagu Leonard Jenyns, with whom he had many discussions about nature, Darwin writes, “At first I dislike him from his somewhat grim and sarcastic expression, and it not often that a first impression is los but I was completely mistaken….” The connection between “it is not often that a first impression lost” and evolution will be discussed further on in the book.

Although Darwin shared his thoughts on evolution and the ample evidence he had found supporti his theory with his scientific colleagues, among them some of the best-known English scientists those days, he refused to publish his findings. He agreed to publish his theory only after Alfre Wallace, a young naturalist researcher who had undertaken many voyages to South America and th Far East, submitted a paper for publication containing ideas similar to those of Darwin but with only flimsy basis. Darwin's friends became aware of this and urged him to publish his book, *The Origin Species*. As a result, in the first official presentation of the theory, Wallace's article and Darwin theory appeared at the same time.

There were several reasons for Darwin's long hesitation before publishing his conclusions. Son derived from the possible conflict with the religious belief that different species exist because that how they were created. Darwin's wife, Emma (née Wedgwood), was deeply religious, and Darwin di not wish to upset her. Another reason, however, no less important, for his hesitation to publish h mechanism of evolution was that despite the wealth of facts he had available supporting his theory evolution, many aspects of the theory had not yet been demonstrated and lacked a scientific basis. particular, Darwin could not offer a biological mechanism that would cause mutation. Th mechanism was not discovered until the middle of the twentieth century, when the genes encoded DNA molecules were revealed and were found to mutate randomly in the course of reproduction.

Mutation and selection of genes are the basis of the modern understanding of the process of evoluti of plant and animal species. The genes carry the characteristics vital to the survival and developme of each species. The ensemble of genes and the way they are expressed, at times as a reaction to th environment, define the characteristics of a species. Changes in the genes are responsible for chang in the species. Nevertheless, much can be learned about evolution without monitoring the changes the genes themselves. By studying the conditions in which the species developed, survived, and w vict~~orious in the evolutionary struggle, we can learn about the characteristics that are encoded~~ in i genes and are passed on from generation to generation. The reverse of this claim is also correct: t characteristics observable at any given time enable us to learn about the conditions in which ea species developed.

The following example shows how we can learn about the link between the conditions in whic species developed and their characteristics today. I came across this example in the course of a trip made to the Galapagos Islands a few years ago. It relates to the mating habits of birds.

The rock cormorant in the picture on the left cannot fly. It lives on cliffs close to the seashore th are exposed to strong winds. Its abilities to find twigs and to build a proper nest are vital to t survival of the species in these tough conditions. In its courtship, the male cormorant demonstrates his potential mate his ability to gather twigs to build the nest they would share. The courted fema will respond to the male's advances only if he can prove he has that capability. The middle picture of the frigate bird. In its courting display, the male inflates his gular pouch until it becomes enormous red balloon. He does this to show his intended mate the strength of his lungs and his abili to fly long distances to scoop fish out of the water. The picture on the right, the blue-footed boob demonstrates entirely different qualities in its courtship display. The male of this species incubat the eggs and protects them by covering them with his large blue feet. He therefore tries to woo h potential mate by flaunting the size and shape of its feet, thus proving his ability to protect the eg they will produce together against enemies and unsettled or rough weather.

These examples illustrate that the characteristics and behavioral patterns we can discern tod indicate the characteristics that were of evolutionary importance, and those show us how each speci survived the evolutionary struggle.

The essential characteristics that helped any particular population to win the battle for surviv during the formation of the species are etched into its genes, and we may identify them as inna attributes. Cheetahs’ speed, eagles’ “eagle eyes” and cats’ tree-climbing ability are all innate traits. cheetah cub is born with the ability and basic instinct to run fast. It will need help from its parents learn what it needs to fear, how to hunt, and even how to run more effectively, but the basic features speed and hunting are carried in its genes. Similarly, the genes of a cat enable it to learn how to cat mice, and the innate characteristics of an eagle include its keen vision and ability to identify potenti prey from a great height. Learning merely refines and improves the innate attributes. The attributes each species enable us to learn about the conditions in which the species developed; similarl knowing the conditions in which the species developed enables us to learn about the characteristi that evolved.

It is reasonable to assume that just as physical attributes of animal species are innate featur etched into their genes, the same will apply to at least some mental attributes. Mental and social skil also play a role in the battle for survival, so that in these too, the selection process strengthens t features that help the species to overcome its rivals. Specifically, in the reproduction process ment attributes can also be changed and improved by mutation. In the following sections we will exami the mathematical capabilities of the human species from an evolutionary aspect. We will ask whethe the ~~abilities to understand and to use mathematics are the results of evolutionary developm~~ent, whether they may be by-products of a brain that developed to cope with other needs.

2. MATHEMATICAL ABILITY IN THE ANIMAL WORLD

If mathematical ability made a contribution in the evolutionary struggle that brought the human ra to the position it currently occupies among the species, it may be assumed that other living bein would possess a certain degree of mathematical ability. But what does mathematical ability mea Mathematics encompasses a broad range of topics and conceptual methods. The question to as therefore, is which of those mathematical features provide an evolutionary advantage? And t follow-up question is how can we identify these mathematical abilities in animals?

The most basic mathematical ability is counting. It is followed by the understanding of the conce of a number as an abstract object and the ability to perform simple arithmetic operations, such addition and subtraction. We will start by discussing the existence of these simple elements in adu animals. A mother cat moves her kittens from place to place and generally does not forget a kitten two, and when she has finished moving them, she does not usually go back again to check whether s has moved all of them. She may remember them individually, but it seems reasonable to state that t mother cat has a sense of quantity. The instinct of quantitative estimation clearly provides evolutionary advantage, so we should not be surprised that adult animals possess that ability. But do that ability extend to the ability to count and to the possibility of performing arithmeti manipulations?

Before presenting several convincing examples showing that some animal species do ha mathematical ability, a warning is in order. The results of experiments in general, and of anim experiments in particular, should be interpreted with great caution. A well-known illustration of this the case of “Clever Hans.” (More details and references concerning this story, and concerning t research mentioned later in this section, can be found in the monographs by Dehaene and by Dev [2000] listed in the sources.) Toward the end of the nineteenth century a horse known as Clever Ha was exhibited on tour in Germany with its trainer, Wilhelm von Osten. The horse showed remarka ability in adding, subtracting, finding the squares of numbers, simple division, and so on, all with very high degree of success. The horse was wrong occasionally, but such errors occurred infrequent The method by which the horse showed its abilities was that when an exercise was read out or writt on a board, it would tap its hoof the number of times corresponding with the right answer. It w suspected that the act was simply a clever deception by which the trainer somehow or other manag to give the right answer to the horse. An official committee was appointed, headed by a psycholog named Carl Stumpf and whose members included the director of the Berlin Zoo. The committ checked, among other things, whether the horse could solve the problems if the trainer was n present, and it found that even then the horse could still give the correct answers. The conclusion w that some animals have a fairly advanced mathematical ability. Subsequently, more detail examinations in 1907 by another psychologist, named Oscar Pfungst, showed that the horse did n know mathematics. The trainer was indeed reliable and honest, but the horse had learned distinguish involuntary changes in his facial expressions and in the facial expressions of the audien when the trainer was not present. The horse understood from those facial expressions when it h reached the correct number of taps of its hoof. The presence of the trainer or an audience w essential. Pfungst found that if the trainer looked tense at a wrong answer, the horse answer according to the expression and not the correct answer. The research methods developed by Pfungst a result of this case are now recognized as a breakthrough in psychological research.

~~Scientific experiments that were more soundly based have proven that some animals do~~ indee possess mathematical ability. The German zoologist Otto Koehler (1889–1974) proved as early as the 1930s that some species of birds can identify a collection with a given number of elements. It apparently not difficult to train a pigeon to choose every third seed when faced with a row of seeds. squirrel can be trained so that when faced with boxes containing different quantities of nuts, it wi choose the box with exactly five nuts. There is a limit to the numerical-identification ability of the animals. Koehler himself found that even the most capable animals could not identify collections wi more than seven elements. The number appears in the literature also as a bound to the number information units that a human brain can process. We will meet the number seven again later on similar contexts. Still, these experiments demonstrate the mathematical ability to estimate quanti but do not yet prove an ability to count or to grasp the abstract concept of a number.

Adult crows are known to be able to count, within certain limitations. Food is placed near building. The crow learns very quickly that it is dangerous to attempt to approach the food whi someone is in the building. It cannot see into the building to check if anyone is inside or not, but it ca see when someone enters or leaves it. The popular literature (without scientific checks, it must said) reports situations in which several people enter the building one after the other. As long as the remain in the building, the crow keeps away. The people in the building then leave, one by one. Wi surprising accuracy the crow knows when all those it saw enter the building have left, and only the does it approach the food. Clearly there is a limit to crows' ability to be exact, just as there is a lim to humans' ability to keep track on exact large numbers. Crows managed to count up to five or six this manner, with a high degree of accuracy. The ability to identify a collection with a given numbe of elements demonstrated by crows in this example and by other species is consistent with evolutionary advantage.

The ability to count is clearly an advantage in the battle for survival, but its origin in the avia world is unclear. After all, how often in the evolution of crows did they encounter a situation in whic they had to count the number of dangerous animals entering and leaving a building? Specifically, it unclear whether this apparent counting is in fact counting in the mathematical sense. In other word does the crow have the ability, whether conscious or not, to comprehend the number of the peop entering the building, or does it simply remember who went in and who came out?

Monkeys were found to have a greater mathematical ability to count and compare. The followi experiments were carried out by Guy Woodruff and David Premack of the University of Pennsylvan (their paper was published in 1981). A chimpanzee was shown a full glass and a half-full glass, and was taught to choose the half-full glass every time. The same chimpanzee was then offered the choi of a whole apple or half an apple, and it chose the half apple. In other words, it generalized th mathematical principle from the glass to the apple. In a similar fashion, the chimpanzee was taught demonstrate simple mathematical abilities, such as recognizing that the combination of half an app and a quarter of an apple is three-quarters of an apple. In another experiment, two trays were plac before a chimpanzee. The first tray had two piles of pieces of chocolate, one pile with three piece and the other with four. The second tray had a pile of five pieces of chocolate and then a separat single piece. In most cases, the chimpanzee chose the tray with the larger total number of pieces. Th does not yet constitute proof that the chimpanzee understood the abstract concept of numbers or th addition of numbers, but it is evidence of mathematical abilities. This is not surprising, as suc abilities constitute an evolutionary advantage.

Another experiment with animals proves that the concept of numbers in the abstract does exist some degree among some, even among less-developed animals. The experiments were conducted Russell Church and Warren Meck of Brown University (the research was published in 1984). It is n difficult to train rats so that when they hear two beeps, one after the other, they are given enough tas ~~food to satisfy them. Similarly, when they see two flashes of light, they can also safely eat the~~ foo They were taught, however, that when they hear four beeps or see four light flashes, it is dangerous eat the food, as they get an electric shock. The aural or visual signals, that is, the beeps or flashes, a received and processed in the brain via two different senses, hearing and sight. The rats reached a hi level of reacting correctly, approaching the food if they heard two beeps or saw two flashes, a avoided doing so if they heard four beeps or saw four flashes. When the rats had been train sufficiently, they heard two beeps that were immediately followed by two light flashes. How do y think they reacted? Did the rats consider the signals as a double invitation to eat the food, or did th interpret them as a four-signal warning to refrain? If they reacted according to the latter, it may assumed that they recognized the number four as an independent concept, even though the signa received were of two different types. The answer: the rats clearly identified the number four and d not approach the food when they received four signals, even when they were received via differe senses.

This experiment with rats still does not indicate arithmetic ability in these animals, nor does it give definite proof that such abstract counting is an innate attribute, that is, a characteristic carried in the genes, as it may be the result of training made possible by the development of the brain for oth purposes. It seems reasonable, however, that this ability is innate, mainly because of the evolutiona advantage given by the abilities to count and to recognize the concept of numbers. To be convince beyond all doubt that a particular ability is innate, it should be identified in the animal when it is st very young. Such experiments with cubs and other animal young are obviously very difficult perform. With human cubs, that is, babies, such experiments can be performed.

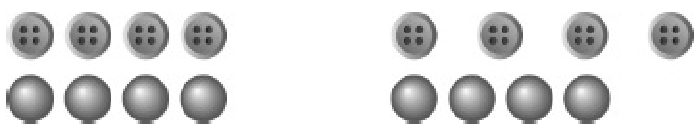
3. MATHEMATICAL ABILITY IN HUMANS

Before we present the evidence that mathematical ability is inborn in human beings, that is, embedd in their genes, we need to make two comments about the nature of the discussion. First, our use of t term *genes* from here on is a conceptual one and does not relate to any specific gene or set of gene We will leave the identification of the genes responsible for mathematical ability to our biologi colleagues. For us, establishing the fact that it is innate is sufficient. Second, in the examples animals and in the discussion in this section, we do not relate to the ability of any individuals. We not ask whether the success in mathematics of a specific student is determined by his genes alone whether it is due to environmental conditions or to his having good or less-good mathemati teachers. The discussion is concerned with the mathematical capabilities of the human race and t connection between that capability and the process of evolution, a process that has continued f millions of years, in the course of which the abilities under discussion were formed.

We will first address the simplest mathematical operations, that is, counting, addition, an subtraction. One of the basic principles of classical psychology is that babies are born with a brain th evolution has prepared for learning but that is initially void of all information. Babies learn about t world initially via observation and then via a combination of observation and experience. Mor abstract learning appears later, with language development. This view was held and taught by no le than Sigmund Freud (1856–1939), the father of modern psychology. What he said related knowledge in general and to mathematical ability. At first glance it does appear that regardi mathematical elements that description is correct. Only when they are three or four years old children acquire the ability to count, and later on to add and subtract. At first they only recite wh they have heard, one, two, three, and so on, without realizing that they can count. To illustrate, if give ~~three balls, they may count one, two, three, four, five, counting the same ball more than once.~~ Only a later age do children begin to understand what counting is, and even later do they start performin simple arithmetic operations. A leading researcher and advocate of this approach was the famou psychologist Jean Piaget (1896–1980), who formulated a complete theory of cognitive developme regarding the gradual acquisition of mathematical abilities from childhood to adulthood. (The read can find elaborations on this issue and other research referred to later in this section in th monographs by Dehaene and Devlin [2000] listed in our sources.) In one of his experiments, Piag showed children eight flowers, six roses and two chrysanthemums, and asked, “Are there more flowe or more roses?” A significant number of children answered roses. Piaget concluded that children hav no intuition of set inclusion; in other words, the children had no understanding that with two sets, o of which includes the other—in our case the set of flowers includes the set of roses—the former larger. In Piaget's time, it was believed that the relation between sets was the right basis f mathematics (a view that is currently becoming less and less accepted; more on this in the last chapt of the book). Accordingly, Piaget concluded that small children have no understanding of th connection between sizes of sets and of one set including another, let alone any ability to count knowledge of simple arithmetic.

Nevertheless, the fact that the ability to count is not acquired until a child is several years old do not necessarily prove that the characteristic is not innate. In the observations mentioned abov including Piaget's experiments, counting and arithmetic were acquired together with the ability communicate and to use a given language, generally the mother tongue. It is not surprising th communicating in a given language is not an innate attribute but a learned one. The ability to learn language is an inborn characteristic, but acquiring the language itself takes several years. Before th child learns a language, his arithmetic abilities do not come into play, as seen in the abov experiments. The obstacle is not the child's lack of ability to count but the fact that he has to answ questions that he does not comprehend, or he does not realize what the expected answer is, until he h had some more years of practice. It is easy to devise tests showing that understanding the questio plays an important role in interpreting the results.

Children aged three to four years are shown four marbles and, nearby, four buttons; they are aske whether there are more marbles or more buttons. Most of them will answer that the number of marbl and buttons is the same. The buttons are then spread more widely, that is, spaced farther apart fro each other, and again the question is asked, “Now, are there more marbles or more buttons?” Most the young children will give the same answer, the number is the same. When this exercise is repeate with older children, aged five and six years, many of them answer that there are more buttons.



This does not indicate a drop in their mathematical aptitude. The correct explanation is that old children are not used to being asked the same trivial question more than once. They therefore conclu that the questioner must expect a different answer and assume that the question is about the distan between the items and not their number, and they answer accordingly.

There is clear evidence that very young infants relate to numbers and can even perform simp addition and subtraction. How can such cognitive capabilities be examined in babies only a fe months old? Several parameters enable us to see whether a baby is excited or surprised. One is th length of time it looks at something. A baby can look at an object or a situation for a few seconds ar then it will divert its gaze to something else. When it looks at something new or surprising—and f bab~~ies a few months old, new is also surprising—it holds its gaze on it for a longer period, for~~ a fe seconds more. A second parameter is the rate at which the baby sucks, say its pacifier. When it excited or surprised, it sucks harder and more frequently.

An experiment undertaken by Ranka Bijeljac-Babic and colleagues in Paris (the results of whic were published in 1991) showed that even newborn babies have a sense of numbers. They measure the intensity with which babies sucked while hearing meaningless words of three syllables, such “defantok,” “alovo,” “kamkeman.” At first, when the babies heard the words, they sucked harder unt they became accustomed to the sounds, and then they reverted to normal sucking. Then two-syllab words were spoken, and this led to harder sucking again. This pattern repeated itself. Whenever th number of syllables in a series of words changed, the reaction of the babies changed too. In oth words, even at such an early age, babies can recognize that word sounds consist of syllables and rea to a change in the number of syllables between one series and another. The syllables in the “word were chosen randomly to avoid their having any meaning or significance, so that the only explanatio for the babies’ reaction was the number of syllables.

A more complex experiment performed in the laboratory of Prentice Starkey of the University Pennsylvania (the results of which were published in 1980) showed that distinguishing betwe different numbers is not restricted to one communication channel. Six-month-old babies were show pairs of pictures with either two or three elements, say two in the picture on the left and three in th picture on the right. Different objects were shown each time, sometimes just geometric shape sometimes dots, and so on, and each time the colors were different; this was done to neutralize an possible effect of the content of the pictures. While the pictures were being shown the babies als heard notes or sounds, sometimes two and sometimes three, in random order, in order to cancel an possible effect of any structure in the order in which the notes were heard. When three notes we heard, the babies clearly preferred to look at the picture with the three images, and when they hea two notes, they turned their attention to the picture with two elements. They were exhibiting counting operation or were at least comparing quantities perceived via two different senses, sight an hearing.

Another sophisticated experiment conducted by Karen Wynn of Yale University (results publishe in 1992) showed that babies have a natural sense of addition and subtraction. A screen was place before babies of a few months, and they saw a figure going behind it. The screen was removed, an they saw the figure. Next, one figure went behind the screen, and then another figure followed. Th screen was removed, and the babies saw the two figures. This was repeated several times until th babies became accustomed to what was happening. Then an arithmetically incorrect exercise w performed. One figure went behind the screen, and then a second figure. When the screen w removed, only one figure could be seen. To a highly significant degree, the several-months-old babi showed surprise. They expected two figures, and lo and behold, there was only one! The experime was repeated with a number of variations to remove the possibility that the infants were simply use to the result of a particular exercise. It was highly significant that results that were arithmetical incorrect gained more of the babies’ attention. Later, a similar experiment was conducted with adu rhesus monkeys. They showed signs of surprise when, for example, a banana was put inside a bo followed by a second one, and when the box was opened, there was only one banana inside.

These experiments were scrupulously and rigorously controlled, and it may be concluded from the that human beings’ arithmetic abilities are genetic. Clearly these operations are performed undeveloped brains and in no particular language, and there is thus no possibility for the baby discuss the results with its parents or friends. When the child grows, it will have to learn how express this mathematical ability in the everyday language it uses to talk to its parents. This learnin is a process in itself. But simple arithmetic is innate in babies and is not a by-product of a brain th had ~~been developed for completely different purposes. From this it may be deduced that~~ simp arithmetic afforded an advantage in the evolutionary competition. This is not surprising. For tho competing for food, the mathematical ability to distinguish large from small, the many from the fe and even addition and subtraction gives an evolutionary advantage. An individual with this ability wi be better suited to a competitive environment than would other members of the same species wi lower mathematical abilities.

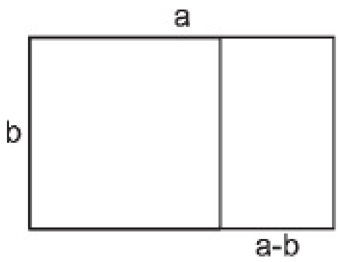
How is this finding consistent with the finding that some primitive tribes, including son discovered recently in isolated locations, use only the numbers one, two, and three to describe the environment, and any larger quantities are referred to as “many”? If living beings such as birds or ra can differentiate between numbers greater than three, one would expect humans to be able to cou better. The answer is simple: language developed much later among human beings in the process evolution and placed greater emphasis on more important things than the less important. Tho primitive tribes apparently are well aware of the difference between sets consisting of five or s objects, but their language is not rich enough to describe them because they had no need to devo terms to numbers greater than three. This does not contradict the fact that at the intuitive level the arithmetic capability is much higher. As a language develops, so does the ability to express an perform more-extensive arithmetic operations. Language developed relatively late in the gener evolutionary process but is itself part of that process. The human brain is distinctive among livi beings in its verbal communication abilities. Indirect evidence that arithmetic, counting, and th facility to perform addition and subtraction, for example, are the direct results of evolution and n just by-products of language can be found in recorded cases of people born with a malfunction of th brain that meant they could not count or perform addition or subtraction but whose other verb capabilities were perfectly normal. Conversely, there are people with defective verbal capabilities wh can perform arithmetic operations easily.

It should be noted that similar techniques of research into the evolutionary roots of mathemati can be used to discover abilities and features whose roots are evolutionary and that are unrelated mathematics. Recently (in 2010) research by Karen Wynn of Yale, mentioned above, and her partne Paul Blum was published, showing that altruism and the aspiration for justice exist in babies a fe months old, at an age when it is reasonable to assume they could not have absorbed the characteristics from the environment. This too is not surprising. The preference for a just distributio of resources is an attribute that helps a society survive the evolutionary struggle, and it is reasonabl therefore, that it is inherent at the genetic level.

4. MATHEMATICS THAT YIELDS AN EVOLUTIONARY ADVANTAGE

Mathematics has many aspects. The previous section showed that the ability to perform arithmet calculations is the result of evolution. In this section we will indicate other branches of mathematic operations that, it may reasonably be assumed, provided an advantage in the evolutionary struggle. W will present evidence that those parts of mathematics were also incorporated in the genetic heritag We may refer to this aspect of mathematics as *natural mathematics*. In the next section we wi describe mathematical operations that are not natural, as they did not afford any evolutiona advantage in the hundreds of thousands of years during which the human genome was formed.

It is reasonable to assume that the ability to recognize geometrical elements gave an evolutiona advantage. As sources of food and water have typical geometric shapes, being able to recognize tho shapes correctly constituted an advantage in the competition for sources of sustenance. But is the any ~~evidence that, as a result of evolution, the recognition of geometric shapes is carried by the~~ gene We will soon turn our attention to such evidence but will first introduce what is known as the golde cut, or the golden-ratio rectangle.



The golden-ratio rectangle is one in which the ratio between the longer side and the shorter side such that if a square with sides the length of the shorter side of the rectangle is removed, the sides the rectangle that remains will have the same ratio as the original one. We note, although this is no relevant to our tale, that it is not difficult to calculate the numerical value of the golden ratio (ar skipping the calculation below will not impair the reader's understanding of what follows).

Denote the length of the rectangle by a, and the width by b. The relation required between the ratios is expressed by $\frac{a}{b} = \frac{b}{a-b}$. If w denote by x the desired ratio $\frac{a}{b}$, the unknown x satisfies the quadratic equation $x^2 - x = 1$, the solution of which is (recalli secondary-school mathematics) $\frac{1+\sqrt{5}}{2}$. This is the golden ratio, approximately 1.6180 in decimal numbers.

The golden ratio appears in many instances and processes in nature, and several of its attributes we known in ancient times. It has been identified in ancient architecture. For example, the dimensions the Parthenon in Athens are amazingly close to those defined by the golden ratio. The ratio can also b discerned in Leonardo da Vinci's paintings; he also referred to it in his mathematical writing although he did not state that he used it in his art.

The discovery of the golden ratio in various and sometimes-unexpected forms in nature resulted the ancients attributing mystical properties to it, and they even referred to it as the *divine proportio* For many years, a lively debate continued and is still continuing today among historians and artists o the question of whether builders and artists in ancient times made conscious use of the golden ratio their architecture and art, or whether its frequent appearance is due to the fact that it is aesthetical pleasing. We will not join this open debate at this point but will just note that the ratio is indeed ver pleasing to the eye. This has been proved in dozens of empirical studies, including studies showin that infants react with greater pleasure and calm to golden-ratio rectangles than they do to rectangl with other proportions, including those with greatly different proportions.

This needs to be explained. We are used to the fact that the pleasantness of a drawing or a paintin or a shape to the adult eye is highly dependent on familiarity and education. For example, the attitu toward modern art initially was almost hostile, and it moderated over the years as the general publ became more and more familiar with it. Babies have not had time to become familiar with an particular shape or form. What, then, is the origin of their preference for the golden ratio? The answ is simple: Evolution. An examination of the dimensions of the human head reveals that they are clos to the golden ratio. Likewise, the proportions of sections of the human face, such as the ratio of th width and height of the eyes, the height and the width of the ears, and so on, are also close to th golden ratio. The evolutionary advantage to an infant who can recognize and is happy to discover figure with those proportions is clear. Babies who are calm when they see their mother approaching, contrast to exhibiting discomfort or even crying for help when they see a bird of prey nearby, have ~~greater chance of surviving. Hence the feeling of greater comfort when confronted with rectangul~~ forms that have proportions similar to those of the human face rather than other forms is etched in the human genes. This has nothing to do with the golden ratio itself. In fact, research shows th babies also feel at ease with the shape of a hand, and the evolutionary reason for that is self-evider Evolution rewards a baby who reacts with discomfort if held by a predator compared to its reactio when held by a human being. I would hazard a guess that if it were possible to perform simil experiments with birds, we would find that of all geometric forms, the most pleasing to a young chic would be an acute-angled triangle.

At this stage we may still wonder whether babies may have learned to feel comfortable with ratio similar to the golden ratio in the first weeks after their birth. The answer lies in the signs discomfort and fear when they are faced with certain forms. Psychologists claim that about one-ten of all children experience a primeval fear of clowns. Recently an occupation known as medic clowning has become widespread. It involves clowning activities meant to relax and help childre requiring hospitalization. But cases have also been reported in which the activities of the medic clown only harmed the child, and the condition of the terrified children deteriorated when they saw th clown. This too is related to geometry and its evolutionary roots. The sight of a clown, with all h bright colors and the nonhuman proportion of his limbs and head, calls into play the same genes th make infants cry for their parents' help when they see a multicolored bird of prey or a tig approaching. It is unreasonable to think that in the modern world children would "learn" to be afra of clowns. These innate features are the inception of geometric recognition. (We will often refe below to the simple but illustrative metaphor of confrontation with a tiger.)

Another basic mathematical ability that almost certainly played a role in the evolutionary struggle the ability to identify patterns. I am not familiar with controlled experiments that show that th tendency and ability to recognize patterns is ingrained in the genes, but imagine early man with a tig stealthily creeping up on him in the grass, leaving a trail of flattened grass. The ability to identify th trail as a source of danger could be life-saving. Ability to recognize patterns is not restricted to visu patterns. Consider for instance patterns of sound. For most of us, hearing very few notes is enough recognize a pattern and sometimes to identify an entire tune or melody. As recognizing patterns is a attribute that is helpful in the evolutionary struggle, those who had this ability had more offsprin than those that lacked it. It is thus almost certain that the tendency to recognize patterns is passed o genetically. Less harm is caused by seeing a pattern where none exists than by failure to identify a existing pattern. Thus, the evolutionary tendency to identify patterns also results in identifyin ostensible patterns, including patterns that do not exist. We can take the Bible code as an instance o incorrect identification. By constructing sentences consisting of only every *n*th word of the text in th Bible, it can apparently be shown that many of the events in modern life were allegedly foreseen in th Old Testament. Careful statistical tests proved that these patterns have no scientific reality. From th outset, however, the tendency to find patterns overcame scientific caution. In later sections we wi come across other mental errors deriving from discovering patterns where they do not exist.

Much of mathematics, both in research and in the various stages of learning mathematics, focus on the identification of patterns in sequences. Here are a few simple exercises. Continue the sequenc

$$2, 4, 6, 8, 10,\ldots$$

At a relatively early age, children will recognize the sequence of even numbers and will correctly giv the next numbers in the sequence, 12 and 14. More knowledge is required to recognize the followin sequence:

1, 4, 9, 16, 25, 36,…,

but it is not difficult to see that the numbers in the sequence are the squares of the numbers 1, 2, 3, 5, 6, so that the following numbers will be 49 and 64. We should point out and emphasize that thes sequences do not necessarily continue as we have suggested. In other words, these extensions of th sequences do not derive from a logical necessity. Moreover, the answers are culture dependent. He is an exercise attributed to the mathematician and historian Morris Kline. Continue the sequence:

4, 14, 23, 34, 42, 50, 59,…

The answer? 72. The numbers in the sequence are the numbers of the streets at which the Manhatta Subway C stops, and the next station is at 72nd Street. I would guess that if regular travelers on th New York subway were given this exercise, many would have given the answer 72. I have deliberatel avoided saying that they would have given the right answer, because this is not a matter of right an wrong. The answer is right if that is what the questioner intended. It is easy to see, however, that th human race has the inborn intuition to continue series such as the above in a reasonable manner, an to understand what the questioner wants. (We will discuss again this exercise in the last chapter of th book.)

Clearly one must not exaggerate, and the story of the four-engine airplane flying from New York t London comes to mind. About an hour after takeoff, the pilot announces that one of the four engin has failed, but there is nothing to worry about. The other three are working as they should, and th flight would just take nine hours instead of the originally scheduled six. A short while later the pil announces that a second engine had ceased functioning, but not to worry, the only effect was that th flight would now take twelve hours. A while later comes the third announcement, that the third engi is now out of action, so the flight time is now fifteen hours. At this point a passenger jumps up an asks, “Is there enough food and drink on board in case the fourth engine fails and the flight tak eighteen hours?” (It would be interesting to ask mathematics students to complete the sequence in th event that the fourth engine stopped working.)

Some continuations of sequences, even if there is no logical necessity, are directly connected wi natural phenomena. Let us take, for example, the following sequence:

1, 1, 2, 3, 5, 8, 13, 21,…

Each number (from the third) in the sequence is the sum of the previous two numbers, so that the ne two in the sequence would be 34 and 55, and so on. This is the Fibonacci sequence, named after th Italian mathematician Leonardo Fibonacci, or Leonardo of Pisa (1170–1250), whose book *Liber Aba* (1202) included extensive development of the properties of this sequence. It reflects many aspects development and growth in nature, as well as mathematical properties that are interesting themselves. We describe one use of the sequence here.

Certain trees, including some types of mangrove, increase in number by a branch taking root in th ground and growing into a new trunk. A year has to pass, however, until a branch of a young mangrov can send out one of its branches from which a new tree will grow. Assume that a young mangrove planted in the ground. After one year there will still be one mangrove tree, but after two years a branc of the first tree will also be growing, so there will be two mangroves. This is the beginning of th sequence 1, 1, 2. The next year, only the first tree can send out a branch to take root, so in the four year there will be three trees. The year after that, the two oldest mangroves will send out a branc each, so there will be a total of 2 + 3 = 5 trees growing, and we already have the sequence 1, 1, 2, 3, and ~~so on. Each year the number of new trunks is equal to the number of older trees (more than~~ a ye old), and the sequence describing their number of trees is the Fibonacci sequence. We will not expar the scope of this matter beyond the example quoted, but I will just add that if a number in th sequence is divided by the preceding number, the further along the sequence we go, the closer is th result to the golden proportion discussed above. This is another fact that convinced the ancients th they were observing a divine proportion or ratio. The fact that series whose extensions can b discovered intuitively are reflected in natural phenomena boosted the tendency to develop the abili to identify patterns throughout the generations.

We will summarize the observations in this and the previous section by stating that we can point t and to some extent corroborate by means of experiments, mathematical abilities that througho hundreds of thousands of years of evolution afforded an advantage in the evolutionary struggle f survival. The processes of mutation and selection by which evolution shaped the human race result in those abilities being etched into human genes.

5. MATHEMATICS WITH NO EVOLUTIONARY ADVANTAGE

In this section we will examine a number of aspects of mathematics that, apparently, are not carrie by our genes because they did not provide an evolutionary advantage during the formation of th human species (other nonnatural aspects of mathematics will be discussed later on). The curre discussion is speculative, but further on we will present evidence corroborating the observations ma here. We emphasize once again that the lack of an evolutionary advantage we are referring to relat to a period in which the genes determining the human species were developing. That is wh mathematics of the type we will discuss here is not natural to intuitive thinking. This does not mea that this aspect of mathematics is not important or useful. Just the opposite. This type of mathemati ability provides a great advantage in the later evolution of human societies, but the time that h elapsed since human societies developed is not long enough for these abilities to have been etched in their genes.

The language of mathematics makes much use of quantifiers, expressions such as “for every,” “there exists” that appear in mathematical propositions. For example, Pythagoras's famous theore which was proved as early as two thousand five hundred years ago, states that *for every* right-angl triangle, the sum of the squares on the two sides equals the square of the hypotenuse. The emphasis on the quantifier “for every.” Another useful claim states that every positive integer is the product prime numbers. A recent famous example is Fermat's last theorem. The hypothesis that it was corre was formulated as early as the seventeenth century but was unproven until the proof by mathematici Andrew Wiles of Princeton University, which was not published until 1995. The theorem states th for every four natural numbers (i.e., positive integers) X, Y, Z and n, if n is greater than or equal to the sum $X^n + Y^n$ cannot equal Z^n. Throughout the thousands of years of development of mode mathematics, the proof that a particular property *always* holds was considered an achievement.

However, is it natural to examine whether a particular property *always* holds? When somethi occurs repeatedly under certain conditions, does it naturally give rise to the question whether it occu *every* time those conditions hold? Not so. If experience shows that a tiger is a dangerous predator, th conclusion drawn is that if one meets a tiger one should flee or hide. Losing energy or time in abstra thought about whether that particular tiger always devours its prey, or whether every tiger is dangerous predator, would not afford an evolutionary advantage.

Another concept often referred to in mathematics is the concept of infinity. The Greeks proved th ~~there is an infinite number of prime numbers. Is the urge to prove this statement a natural one~~? C observing many elements, is it reasonable to ask whether there is an infinite number of them? Again, think it is not. Imagine ancient man discovering that a certain region is teeming with tigers. Is worthwhile for him to consider whether there is an infinite number of them, or would it be preferab for him to get as far away as possible from that area as quickly as possible? The question “Is there a infinite number of tigers?” and even the question “Are there many more tigers than the large an dangerous number that I have already seen?” are academic questions, which will only harm those wh devote time and energy to them and hence will impair their chances of surviving in the evolutiona struggle.

Another type of claim developed by mathematics is expressed in the reference to facts that *cann* exist. A statement such as “If A does not occur, then B will occur” is commonplace among teachers students, and researchers of mathematics. We will come across many such examples further on. Th way of thinking is also not natural. Activity of the human brain is based on association, on th recollection of things that happened. To base oneself on an event that did not take place may b possible and useful, but does not come easily or intuitively. When you enter a room, you look at wh is in it and devote less thought to what is not there. We should repeat that we are not claiming th searching for an infinite number of mathematical elements, or proving that a certain property alway holds, or relating to the negation of a possibility is an unworthy, unimportant, or uninterestin activity. What we are claiming is that those activities are not natural and that without a mathematic framework that suggests these possibilities, a reasonable person or an untrained student would n intuitively ask those questions.

Another attribute that is not innate in human nature is the need for rigor and precision. Mathemati is proud that a mathematical proof, provided it does not contain an error, is like an absolute trut Mathematics therefore developed techniques of rigorous tests intended to lead to that absolute trut Such an approach cannot have been derived from evolution. Genes do not direct humans to a rigorously to remove any possible doubt. The following anecdote illustrates this convincingly.

A mathematician, a physicist, and a biologist were sitting on a hill in Ireland and looking at th view. Two black sheep wander past them. The biologist says: “Look, the sheep in Ireland are black The physicist corrects him: “There are black sheep in Ireland.” “Absolutely not,” says th mathematician, “In Ireland there are sheep that are black at least on one side.”

Is the mathematician's claim, however rigorous and correct it may be, reasonable and useful in dai life? Of course not. In that sense, life is not mathematics. In life, even in ancient times, it is and w worthwhile and desirable to allow a lack of rigor, and even to allow errors, in order to achiev effectiveness. If a tiger's head can be seen above a bush, a man should not insist on being precise an saying that it has not been proven that the specific tiger has legs, but instead he had best distan himself from there as fast as he can.

We have claimed that the use of quantifiers and the interest in negatives or the reference to facts th cannot exist were not absorbed into the human brain during the evolutionary process and are n intuitive. Indirect evidence supporting this claim may be derived from studies that examined ho many mathematical operations the human brain can perform consecutively. Calculations such addition and subtraction can be performed one after the other almost without limit. A person can b asked to perform a long series of multiplications, additions, division, and so on, and if he manages remember the order, for instance by discovering a pattern in it, he can internalize the instructions an develop intuition regarding the next operation. This does not apply to quantifiers and negation “Every dog has a collar that is not green.” That statement uses three concepts of logic: *every*; *has*; an *is not*. Studies have shown that even if someone can remember the order of the operations, the large ~~number of quantifiers that the brain can absorb is seven. Beyond that, even the most capable~~ perso cannot assess the outcome of the operation. It is interesting that the limit to the number of logic operations the human brain can absorb is seven, the same number as the maximum number elements that animals can identify (see section 2 above). Other indirect evidence is provided by th existence of certain individuals, some of them autistic and some with Asperger's syndrome, who ca perform complex arithmetic calculations with amazing speed and accuracy. However no individua have been found who can similarly perform complex logical operations. The reason is apparently th the ability to perform arithmetic calculations exists in the brain naturally and is strengthene disproportionately in people whose limitations do not allow them to develop other abilities. Logic not one of those extreme abilities.

Why is it important to identify mathematical abilities that are innate by virtue of evolution and identify other attributes that are not innate? Humans think intuitively, associatively, and it is possib and easy to develop intuition based on natural abilities. Abilities contained in the genes are easier develop, nurture, and use. It is harder to do that with abilities that are not natural to the human specie The recognition that there is a distinction between those two types of mathematical operations an understanding the source of that distinction are important to the understanding and utilization human thought. In the sections that follow, we will see how these differences are significant to th development of mathematics, and in the last chapter of the book, we will discuss the implications recognizing these differences for teaching mathematics.

6. MATHEMATICS IN EARLY CIVILIZATIONS

In this section we will review the mathematics that developed in the Babylonian, Assyrian, an Egyptian kingdoms. We will also look at the mathematics that developed independently and somewh later in the Chinese dynasties. Although this survey does not cover the mathematics created in thos realms exhaustively, it does correctly reflect the type of mathematics that developed. In particular, w will see that its development clearly traces what we have called the evolutionary advantage. The advantages of mathematics not only afforded humans an advantage over other living beings, but the also gave advantages to societies that developed mathematics over others that did not. The societi that ruled were those that developed the most up-to-date mathematics and that used it to establish an expand their power.

Reference to numbers and arithmetic existed prior to the Babylonian and Egyptian civilizations, b there is no direct evidence about where this mathematics existed or its level of development. Based those remote tribes discovered in the last few centuries whose languages included only the numbers 2, 3, many, we may assume that the mathematics these remote tribes used was minimal. In contrast, 1960 human bones were discovered in the Belgian Congo that were dated to 20,000 BCE, and on the were signs that archaeologists and anthropologists believe express counting up to and beyond th number twenty. Thus we may conclude that when man lived and developed in small groups, wa nomadic, and subsisted mainly by hunting, he used and even developed simple mathematics, which w referred to in previous sections as giving an evolutionary advantage.

The Babylonian kingdom was a mighty one, certainly for its time. Its origins date back to 4700 BC Its culture was based on Sumerian culture. Later, the Akkadian civilization became predominan leading to cultural, economic, and social progress. The Akkadian contribution is attributed mainly King Hammurabi, who ruled around 1750 BCE and who is famous mainly through the Code