Fourier Analysis

Javier Duoandikoetxea

Translated and revised by
David Cruz-Uribe, SFO

Graduate Studies in Mathematics
Volume 29

American Mathematical Society

Fourier Analysis

Javier Duoandikoetxea

Translated and revised by

David Cruz-Uribe, SFO

Graduate Studies
in Mathematics

Volume 29

American Mathematical Society
Providence, Rhode Island

ANÁLISIS DE FOURIER

by Javier Duoandikoetxea Zuazo

Published in Spanish by Addison-Wesley and Universidad Autónoma de Madrid in 1995

Translated from the Spanish by David Cruz-Uribe, SFO

2000 *Mathematics Subject Classification.* Primary 42B15, 42B20, 42B25.

ABSTRACT. The purpose of this book is to develop Fourier analysis using the real variable methods introduced by A. P. Calderón and A. Zygmund. It begins by reviewing the theory of Fourier series and integrals, and introduces the Hardy-Littlewood maximal function. It then treats the Hilbert transform and its higher dimensional analogues, singular integrals. In subsequent chapters it discusses some more recent topics: H^1 and BMO, weighted norm inequalities, Littlewood-Paley theory, and the $T1$ theorem. At the end of each chapter are extensive references and notes on additional results.

Library of Congress Cataloging-in-Publication Data

Duoandikoetxea, Zuazo, Javier.
[Análisis de Fourier. English]
Fourier analysis / Javier Duoandikoetxea ; translated and revised by David Cruz-Uribe.
p. cm. — (Graduate studies in mathematics ; v. 29)
Includes bibliographical references and index.
ISBN 0-8218-2172-5
1. Fourier analysis. I. Title. II. Series.
QA403.5 .D8313 2000
515'.2433—dc21 00-064301

∞ The paper used in this book is acid-free and falls within the guidelines
established to ensure permanence and durability.
Visit the AMS home page at URL: `http://www.ams.org/`

10 9 8 7 6 5 4 3 2 1 06 05 04 03 02 01

Dedicated to the memory of
José Luis Rubio de Francia, my teacher and friend,
who would have written a much better book than I have

Contents

Preface

Fourier Analysis is a large branch of mathematics whose point of departure is the study of Fourier series and integrals. However, it encompasses a variety of perspectives and techniques, and so many different introductions with that title are possible. The goal of this book is to study the real variable methods introduced into Fourier analysis by A. P. Calderón and A. Zygmund in the 1950's.

We begin in Chapter 1 with a review of Fourier series and integrals, and then in Chapters 2 and 3 we introduce two operators which are basic to the field: the Hardy-Littlewood maximal function and the Hilbert transform. Even though they appeared before the techniques of Calderón and Zygmund, we treat these operators from their point of view. The goal of these techniques is to enable the study of analogs of the Hilbert transform in higher dimensions; these are of great interest in applications. Such operators are known as singular integrals and are discussed in Chapters 4 and 5 along with their modern generalizations. We next consider two of the many contributions to the field which appeared in the 1970's. In Chapter 6 we study the relationship between H^1, BMO and singular integrals, and in Chapter 7 we present the elementary theory of weighted norm inequalities. In Chapter 8 we discuss Littlewood-Paley theory; its origins date back to the 1930's, but it has had extensive later development which includes a number of applications. Those presented in this chapter are useful in the study of Fourier multipliers, which also uses the theory of weighted inequalities. We end the book with an important result of the 80's, the so-called $T1$ theorem, which has been of crucial importance to the field.

At the end of each chapter there is a section in which we try to give some idea of further results which are not discussed in the text, and give references for the interested reader. A number of books and all the articles cited appear only in these notes; the bibliography at the end of the text is reserved for books which treat in depth the ideas we have presented.

The material in this book comes from a graduate course taught at the Universidad Autónoma de Madrid during the academic year 1988-89. Part of it is based on notes I took as a student in a course taught by José Luis Rubio de Francia at the same university in the fall of 1985. It seemed to have been his intention to write up his course, but he was prevented from doing so by his untimely death. Therefore, I have taken the liberty of using his ideas, which I learned both in his class and in many pleasant conversations in the hallway and at the blackboard, to write this book. Although it is dedicated to his memory, I almost regard it as a joint work. Also, I would like to thank my friends at the Universidad Autónoma de Madrid who encouraged me to teach this course and to write this book.

The book was first published in Spanish in the *Colección de Estudios* of the Universidad Autónoma de Madrid (1991), and then was republished with only some minor typographical corrections in a joint edition of Addison-Wesley/Universidad Autónoma de Madrid (1995). From the very beginning some colleagues suggested that there would be interest in an English translation which I never did. But when Professor David Cruz-Uribe offered to translate the book I immediately accepted. I realized at once that the text could not remain the same because some of the many developments of the last decade had to be included in the informative sections closing each chapter together with a few topics omitted from the first edition. As a consequence, although only minor changes have been introduced to the core of the book, the sections named "Notes and further results" have been considerably expanded to incorporate new topics, results and references.

The task of updating the book would have not been accomplished as it has been without the invaluable contribution of Professor Cruz-Uribe. Apart from reading the text, suggesting changes and clarifying obscure points, he did a great work on expanding the above mentioned notes, finding references and proposing new results to be included. The improvements of this book with respect to the original have certainly been the fruit of our joint work, and I am very grateful to him for sharing with me his knowledge of the subject much beyond the duties of a mere translator.

Javier Duoandikoetxea
Bilbao, June 2000

Acknowledgment: The translator would like to thank the Ford Foundation and the Dean of Faculty at Trinity College for their generous support during the academic year 1998–99. It was during this year-long sabbatical that this project was conceived and the first draft of the translation produced.

Preliminaries

Here we review some notation and basic results, but we assume that they are mostly well known to the reader. For more information, see, for example, Rudin [**14**].

In general we will work in $\mathbb{R}^n$. The Euclidean norm will be denoted by $|\cdot|$. If $x \in \mathbb{R}^n$ and $r > 0$,

$$B(x,r) = \{y \in \mathbb{R}^n : |x-y| < r\}$$

is the ball with center x and radius r. Lebesgue measure in $\mathbb{R}^n$ is denoted by dx and on the unit sphere S^{n-1} in $\mathbb{R}^n$ by $d\sigma$. If E is a subset of $\mathbb{R}^n$, $|E|$ denotes its Lebesgue measure and χ_E its characteristic function: $\chi_E(x) = 1$ if $x \in E$ and 0 if $x \notin E$. The expressions *almost everywhere* or *for almost every x* refer to properties which hold except on a set of measure 0; they are abbreviated by "a.e." and "a.e. x."

If $a = (a_1, \dots, a_n) \in \mathbb{N}^n$ is a multi-index and $f : \mathbb{R}^n \to \mathbb{C}$, then

$$D^a f = \frac{\partial^{|a|} f}{\partial x_1^{a_1} \cdots \partial x_n^{a_n}},$$

where $|a| = a_1 + \cdots + a_n$ and $x^a = x_1^{a_1} \cdots x_n^{a_n}$.

Let (X, μ) be a measure space. $L^p(X,\mu)$, $1 \leq p < \infty$, denotes the Banach space of functions from X to $\mathbb{C}$ whose p-th powers are integrable; the norm of $f \in L^p(X,\mu)$ is

$$\|f\|_p = \left(\int_X |f|^p \, d\mu \right)^{1/p}.$$

$L^\infty(X,\mu)$ denotes the Banach space of essentially bounded functions from X to $\mathbb{C}$; more precisely, functions f such that for some $C > 0$,

$$\mu(\{x \in X : |f(x)| > C\}) = 0.$$

The norm of f, $\|f\|_\infty$, is the infimum of the constants with this property. In general X will be $\mathbb{R}^n$ (or a subset of $\mathbb{R}^n$) and $d\mu = dx$; in this case we often do not give the measure or the space but instead simply write L^p. For general measure spaces we will frequently write $L^p(X)$ instead of $L^p(X,\mu)$; if μ is absolutely continuous and $d\mu = w\,dx$ we will write $L^p(w)$. The conjugate exponent of p is always denoted by p':

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

The triangle inequality on L^p has an integral version which we refer to as Minkowski's integral inequality and which we will use repeatedly. Given measure spaces (X,μ) and (Y,ν) with σ-finite measures, the inequality is

$$\left(\int_X \left|\int_Y f(x,y)\,d\nu(y)\right|^p d\mu(x)\right)^{1/p} \leq \int_Y \left(\int_X |f(x,y)|^p\,d\mu(x)\right)^{1/p} d\nu(y).$$

The convolution of two functions f and g defined on $\mathbb{R}^n$ is given by

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)\,dy = \int_{\mathbb{R}^n} f(x-y)g(y)\,dy$$

whenever this expression makes sense.

The spaces of test functions are $C_c^\infty(\mathbb{R}^n)$, the space of infinitely differentiable functions of compact support, and $\mathcal{S}(\mathbb{R}^n)$, the so-called Schwartz functions. A Schwartz function is an infinitely differentiable function which decreases rapidly at infinity (more precisely, the function and all its derivatives decrease more rapidly than any polynomial increases). Given the appropriate topologies, their duals are the spaces of distributions and tempered distributions. It makes sense to define the convolution of a distribution and a test function as follows: if $T \in C_c^\infty(\mathbb{R}^n)'$ and $f \in C_c^\infty(\mathbb{R}^n)$, then

$$T * f(x) = \langle T, \tau_x \tilde{f}\rangle,$$

where $\tilde{f}(y) = f(-y)$ and $\tau_x f(y) = f(x+y)$. Note that this definition coincides with the previous one if T is a locally integrable function. Similarly, we can take $T \in \mathcal{S}(\mathbb{R}^n)'$ and $f \in \mathcal{S}(\mathbb{R}^n)$. We denote the duality by either $\langle T, f\rangle$ or $T(f)$ without distinction.

References in square brackets are to items in the bibliography at the end of the book.

Finally, we remark that C will denote a positive constant which may be different even in a single chain of inequalities.

Chapter 1

Fourier Series and Integrals

1. Fourier coefficients and series

The problem of representing a function f, defined on (an interval of) $\mathbb{R}$, by a trigonometric series of the form

$$f(x) = \sum_{k=0}^{\infty} a_k \cos(kx) + b_k \sin(kx) \tag{1.1}$$

arises naturally when using the method of separation of variables to solve partial differential equations. This is how J. Fourier arrived at the problem, and he devoted the better part of his *Théorie Analytique de la Chaleur* (1822, results first presented to the Institute de France in 1807) to it. Even earlier, in the middle of the 18th century, Daniel Bernoulli had stated it while trying to solve the problem of a vibrating string, and the formula for the coefficients appeared in an article by L. Euler in 1777.

The right-hand side of (1.1) is a periodic function with period 2π, so f must also have this property. Therefore it will suffice to consider f on an interval of length 2π. Using Euler's identity, $e^{ikx} = \cos(kx) + i\sin(kx)$, we can replace the functions $\sin(kx)$ and $\cos(kx)$ in (1.1) by $\{e^{ikx} : k \in \mathbb{Z}\}$; we will do so from now on. Moreover, we will consider functions with period 1 instead of 2π, so we will modify the system of functions to $\{e^{2\pi ikx} : k \in \mathbb{Z}\}$. Our problem is thus transformed into studying the representation of f by

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi ikx}. \tag{1.2}$$

If we assume, for example, that the series converges uniformly, then by multiplying by $e^{-2\pi imx}$ and integrating term-by-term on $(0,1)$ we get

$$c_m = \int_0^1 f(x)e^{-2\pi imx}\,dx$$

because of the orthogonality relationship

$$\int_0^1 e^{2\pi ikx}e^{-2\pi imx}\,dx = \begin{cases} 0 & \text{if } k \neq m \\ 1 & \text{if } k = m. \end{cases} \tag{1.3}$$

Denote the additive group of the reals modulo 1 (that is $\mathbb{R}/\mathbb{Z}$) by $\mathbb{T}$, the one-dimensional torus. This can also be identified with the unit circle, S^1. Saying that a function is defined on $\mathbb{T}$ is equivalent to saying that it is defined on $\mathbb{R}$ and has period 1. To each function $f \in L^1(\mathbb{T})$ we associate the sequence $\{\hat{f}(k)\}$ of Fourier coefficients of f, defined by

$$\hat{f}(k) = \int_0^1 f(x)e^{-2\pi ikx}\,dx. \tag{1.4}$$

The trigonometric series with these coefficients,

$$\sum_{k=-\infty}^{\infty} \hat{f}(k)e^{2\pi ikx}, \tag{1.5}$$

is called the Fourier series of f.

Our problem now consists in determining when and in what sense the series (1.5) represents the function f.

2. Criteria for pointwise convergence

Denote the N-th symmetric partial sum of the series (1.5) by $S_N f(x)$; that is,

$$S_N f(x) = \sum_{k=-N}^{N} \hat{f}(k)e^{2\pi ikx}.$$

Note that this is also the N-th partial sum of the series when it is written in the form of (1.1).

Our first approach to the problem of representing f by its Fourier series is to determine whether $\lim S_N f(x)$ exists for each x, and if so, whether it is equal to $f(x)$. The first positive result is due to P. G. L. Dirichlet (1829), who proved the following convergence criterion: if f is bounded, piecewise continuous, and has a finite number of maxima and minima, then $\lim S_N f(x)$ exists and is equal to $\frac{1}{2}[f(x+) + f(x-)]$. Jordan's criterion, which we prove below, includes this result as a special case.

In order to study $S_N f(x)$ we need a more manageable expression. Dirichlet wrote the partial sums as follows:

$$\begin{aligned} S_N f(x) &= \sum_{k=-N}^{N} \int_0^1 f(t)e^{-2\pi ikt}\,dt \cdot e^{2\pi ikx} \\ &= \int_0^1 f(t)D_N(x-t)\,dt \\ &= \int_0^1 f(x-t)D_N(t)\,dt, \end{aligned}$$

where D_N is the Dirichlet kernel,

$$D_N(t) = \sum_{k=-N}^{N} e^{2\pi ikt}.$$

If we sum this geometric series we get

$$D_N(t) = \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}. \tag{1.6}$$

This satisfies

$$\int_0^1 D_N(t)\,dt = 1 \quad \text{and} \quad |D_N(t)| \le \frac{1}{\sin(\pi\delta)}, \quad \delta \le |t| \le 1/2.$$

We will prove two criteria for pointwise convergence.

Theorem 1.1 (Dini's Criterion). *If for some x there exists $\delta > 0$ such that*

$$\int_{|t|<\delta} \left| \frac{f(x+t)-f(x)}{t} \right| dt < \infty,$$

then

$$\lim_{N\to\infty} S_N f(x) = f(x).$$

Theorem 1.2 (Jordan's Criterion). *If f is a function of bounded variation in a neighborhood of x, then*

$$\lim_{N\to\infty} S_N f(x) = \frac{1}{2}[f(x+) + f(x-)].$$

At first it may seem surprising that these results are local, since if we modify the function slightly, the Fourier coefficients of f change. Nevertheless, the convergence of a Fourier series is effectively a local property, and if the modifications are made outside of a neighborhood of x, then the behavior of the series at x does not change. This is made precise by the following result.

Theorem 1.3 (Riemann Localization Principle). *If f is zero in a neighborhood of x, then*

$$\lim_{N\to\infty} S_N f(x) = 0.$$

An equivalent formulation of this result is to say that if two functions agree in a neighborhood of x, then their Fourier series behave in the same way at x.

From the definition of Fourier coefficients (1.4) it follows immediately that

$$|\hat{f}(k)| \le \|f\|_1,$$

but a sharper estimate is true which we will use to prove the preceding results.

Lemma 1.4 (Riemann-Lebesgue). *If $f \in L^1(\mathbb{T})$ then*

$$\lim_{|k|\to\infty} \hat{f}(k) = 0.$$

Proof. Since $e^{2\pi ix}$ has period 1,

$$\begin{aligned}\hat{f}(k) &= \int_0^1 f(x)e^{-2\pi ikx}\,dx\\ &= -\int_0^1 f(x)e^{-2\pi ik(x+1/2k)}\,dx\\ &= -\int_0^1 f(x-1/2k)e^{-2\pi ikx}\,dx.\end{aligned}$$

Hence,

$$\hat{f}(k) = \frac{1}{2}\int_0^1 [f(x) - f(x-1/2k)]e^{-2\pi ikx}\,dx.$$

If f is continuous, it follows immediately that

$$\lim_{|k|\to\infty} \hat{f}(k) = 0.$$

For arbitrary $f \in L^1(\mathbb{T})$, given $\epsilon > 0$, choose g continuous such that $\|f-g\|_1 < \epsilon/2$ and choose k sufficiently large that $|\hat{g}(k)| < \epsilon/2$. Then

$$|\hat{f}(k)| \le |(f-g)\hat{\ }(k)| + |\hat{g}(k)| \le \|f-g\|_1 + |\hat{g}(k)| < \epsilon.$$

□

Proof of Theorem 1.3. Suppose that $f(t) = 0$ on $(x - \delta, x + \delta)$. Then

$$\begin{aligned} S_N f(x) &= \int_{\delta \le |t| < 1/2} f(x-t) \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}\, dt \\ &= (ge^{\pi i \cdot})^\frown(N) + (ge^{-\pi i \cdot})^\frown(-N), \end{aligned}$$

where

$$g(t) = \frac{f(x-t)}{2i\sin(\pi t)} \chi_{\{\delta \le |t| < 1/2\}}(t)$$

is integrable. By the Riemann-Lebesgue lemma we conclude that

$$\lim_{N\to\infty} S_N f(x) = 0.$$

□

Proof of Theorem 1.1. Since the integral of D_N equals 1,

$$\begin{aligned} S_N f(x) - f(x) &= \int_{-1/2}^{1/2} [f(x-t) - f(x)] \frac{\sin(\pi(2N+1)t)}{\sin(\pi t)}\, dt \\ &= \int_{|t|<\delta} + \int_{\delta \le |t| < 1/2}. \end{aligned}$$

By the Riemann-Lebesgue lemma both of these integrals tend to 0. The second if we argue as in the previous proof, the first since by hypothesis the function

$$\frac{f(x-t) - f(x)}{\sin(\pi t)} \chi_{\{|t|<\delta\}}(t)$$

is integrable. (Recall that if $|t| < \delta$, $\sin(\pi t)$ and πt are equivalent.) □

Proof of Theorem 1.2. Since every function of bounded variation is the difference of two monotonic functions, we may assume that f is monotonic in a neighborhood of x. Since

$$S_N f(x) = \int_{-1/2}^{1/2} f(x-t) D_N(t)\, dt = \int_0^{1/2} [f(x-t) + f(x+t)] D_N(t)\, dt,$$

it will be enough to show that for g monotonic

$$\lim_{N\to\infty} \int_0^{1/2} g(t) D_N(t)\, dt = \frac{1}{2} g(0+).$$

Further, we may assume that $g(0+) = 0$ and that g is increasing to the right of 0. Given $\epsilon > 0$, choose $\delta > 0$ such that $g(t) < \epsilon$ if $0 < t < \delta$. Then

$$\int_0^{1/2} g(t) D_N(t)\, dt = \int_0^{\delta} + \int_{\delta}^{1/2}.$$

Again by the Riemann-Lebesgue lemma, the second integral tends to 0. We apply the second mean value theorem for integrals[1] to the first integral. Then for some ν, $0 < \nu < \delta$,

$$\int_0^\delta g(t)D_N(t)\,dt = g(\delta-)\int_\nu^\delta D_N(t)\,dt.$$

Furthermore,

$$\begin{aligned}
\left|\int_\nu^\delta D_N(t)\,dt\right| &\le \left|\int_\nu^\delta \sin(\pi(2N+1)t)\left(\frac{1}{\sin(\pi t)}-\frac{1}{\pi t}\right)dt\right| \\
&\quad + \left|\int_\nu^\delta \frac{\sin(\pi(2N+1)t)}{\pi t}\,dt\right| \\
&\le \int_\nu^\delta \left|\frac{1}{\sin(\pi t)}-\frac{1}{\pi t}\right|dt + 2\sup_{M>0}\left|\int_0^M \frac{\sin(\pi t)}{t}\,dt\right| \\
&\le C.
\end{aligned}$$

Hence,

$$\left|\int_0^\delta g(t)D_N(t)\,dt\right| \le C\epsilon.$$

□

3. Fourier series of continuous functions

If f satisfies a Lipschitz-type condition in a neighborhood of x, that is, $|f(x+t)-f(x)| \le C|t|^a$ for some $a > 0$, $|t| < \delta$, then Dini's criterion applies to it. However, continuous functions need not satisfy this condition or any other convergence criterion we have seen. This must be the case because of the following result due to P. du Bois-Reymond (1873).

Theorem 1.5. *There exists a continuous function whose Fourier series diverges at a point.*

Du Bois-Reymond constructed a function with this property, but we will show that one exists by applying the uniform boundedness principle, also known as the Banach-Steinhaus theorem.

Lemma 1.6 (Uniform Boundedness Principle). *Let X be a Banach space, Y a normed vector space, and let $\{T_a\}_{a\in A}$ be a family of bounded linear*

[1]If ϕ is continuous and h monotonic on $[a,b]$, then there exists c, $a < c < b$, such that

$$\int_a^b h\phi = h(b-)\int_c^b \phi + h(a+)\int_a^c \phi.$$

operators from X to Y. Then either

$$\sup_a \|T_a\| < \infty$$

or there exists $x \in X$ such that

$$\sup_a \|T_a x\|_Y = \infty.$$

(Recall that the operator norm of T_a is $\|T_a\| = \sup\{\|T_a x\|_Y : \|x\|_X \leq 1\}$.) A proof of this result can be found, for example, in Rudin [**14**, Chapter 5].

Now let $X = C(\mathbb{T})$ with the norm $\|\cdot\|_\infty$ and let $Y = \mathbb{C}$. Define $T_N : X \to Y$ by

$$T_N f = S_N f(0) = \int_{-1/2}^{1/2} f(t) D_N(t)\, dt.$$

Define the Lebesgue numbers L_N by

$$L_N = \int_{-1/2}^{1/2} |D_N(t)|\, dt;$$

it is immediate that $|T_N f| \leq L_N \|f\|_\infty$. $D_N(t)$ has a finite number of zeros so $\operatorname{sgn} D_N(t)$ has a finite number of jump discontinuities. Therefore, by modifying it on a small neighborhood of each discontinuity, we can form a continuous function f such that $\|f\|_\infty = 1$ and $|T_N f| \geq L_N - \epsilon$. Hence, $\|T_N\| = L_N$. Thus if we can prove that $L_N \to \infty$ as $N \to \infty$, then by the uniform boundedness principle there exists a continuous function f such that

$$\limsup_{N\to\infty} |S_N f(0)| = \infty;$$

that is, the Fourier series of f diverges at 0.

Lemma 1.7. $L_N = \dfrac{4}{\pi^2} \log N + O(1)$.

Proof.

$$\begin{aligned} L_N &= 2\int_0^{1/2} \left| \frac{\sin(\pi(2N+1)t)}{\pi t} \right| dt + O(1) \\ &= 2\int_0^{N+1/2} \left| \frac{\sin(\pi t)}{\pi t} \right| dt + O(1) \\ &= 2\sum_{k=0}^{N-1} \int_k^{k+1} \left| \frac{\sin(\pi t)}{\pi t} \right| dt + O(1) \\ &= \frac{2}{\pi} \sum_{k=0}^{N-1} \int_0^1 \frac{|\sin(\pi t)|}{t+k}\, dt + O(1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2}{\pi}\int_0^1 |\sin(\pi t)| \sum_{k=1}^{N-1} \frac{1}{t+k}\, dt + O(1) \\
&= \frac{4}{\pi^2}\log N + O(1).
\end{aligned}
$$

□

4. Convergence in norm

The development of measure theory and L^p spaces led to a new approach to the problem of convergence. We can now ask:

(1) Does $\lim_{N\to\infty} \|S_N f - f\|_p = 0$ for $f \in L^p(\mathbb{T})$?

(2) Does $\lim_{N\to\infty} S_N f(x) = f(x)$ almost everywhere if $f \in L^p(\mathbb{T})$?

We can restate the first question by means the following lemma.

Lemma 1.8. *$S_N f$ converges to f in L^p norm, $1 \le p < \infty$, if and only if there exists C_p independent of N such that*

$$
\|S_N f\|_p \le C_p \|f\|_p. \tag{1.7}
$$

Proof. The necessity of (1.7) follows from the uniform boundedness principle.

To see that it is sufficient, first note that if g is a trigonometric polynomial, then $S_N g = g$ for $N \ge \deg g$. Therefore, since the trigonometric polynomials are dense in L^p (see Corollary 1.11), if $f \in L^p$ we can find a trigonometric polynomial g such that $\|f - g\|_p < \epsilon$, and so for N sufficiently large

$$
\|S_N f - f\|_p \le \|S_N(f-g)\|_p + \|S_N g - g\|_p + \|f - g\|_p \le (C_p + 1)\epsilon.
$$

□

If $1 < p < \infty$, then inequality (1.7) holds, as we will show in Chapter 3. When $p = 1$, the L^1 operator norm of S_N is again L_N, and so by Lemma 1.7 the answer to the first question is no.

When $p = 2$, the functions $\{e^{2\pi ikx}\}$ form an orthonormal system (by (1.3)) which is complete (i.e. an orthonormal basis) by the density of the trigonometric polynomials in L^2. Therefore, we can apply the theory of Hilbert spaces to get the following.

Theorem 1.9. *The mapping $f \mapsto \{\hat{f}(k)\}$ is an isometry from L^2 to ℓ^2, that is,*

$$\|f\|_2^2 = \sum_{k=-\infty}^{\infty} |\hat{f}(k)|^2.$$

Convergence in norm in L^2 follows from this immediately.

The second question is much more difficult. A. Kolmogorov (1926) gave an example of an integrable function whose Fourier series diverges at every point, so the answer is no if $p = 1$. If $f \in L^p$, $1 < p < \infty$, then the Fourier series of f converges almost everywhere. This was shown by L. Carleson (1965, $p = 2$) and R. Hunt (1967, $p > 1$). Until the result by Carleson, the answer was unknown even for f continuous.

5. Summability methods

In order to recover a function f from its Fourier coefficients it would be convenient to find some other method than taking the limit of the partial sums of its Fourier series since, as we have seen, this approach does not always work well.

One such method, Cesàro summability, consists in taking the limit of the arithmetic means of the partial sums. As is well known, if $\lim a_k$ exists then

$$\lim_{k\to\infty} \frac{a_1 + \cdots + a_k}{k}$$

also exists and has the same value.

Define

$$\begin{aligned} \sigma_N f(x) &= \frac{1}{N+1} \sum_{k=0}^{N} S_k f(x) \\ &= \int_0^1 f(t) \frac{1}{N+1} \sum_{k=0}^{N} D_k(x-t)\,dt \\ &= \int_0^1 f(t) F_N(x-t)\,dt, \end{aligned}$$

where F_N is the Fejér kernel,

$$F_N(t) = \frac{1}{N+1} \sum_{k=0}^{N} D_k(t) = \frac{1}{N+1} \left(\frac{\sin(\pi(N+1)t)}{\sin(\pi t)} \right)^2.$$

F_N has the following properties:

$$F_N(t) \geq 0,$$