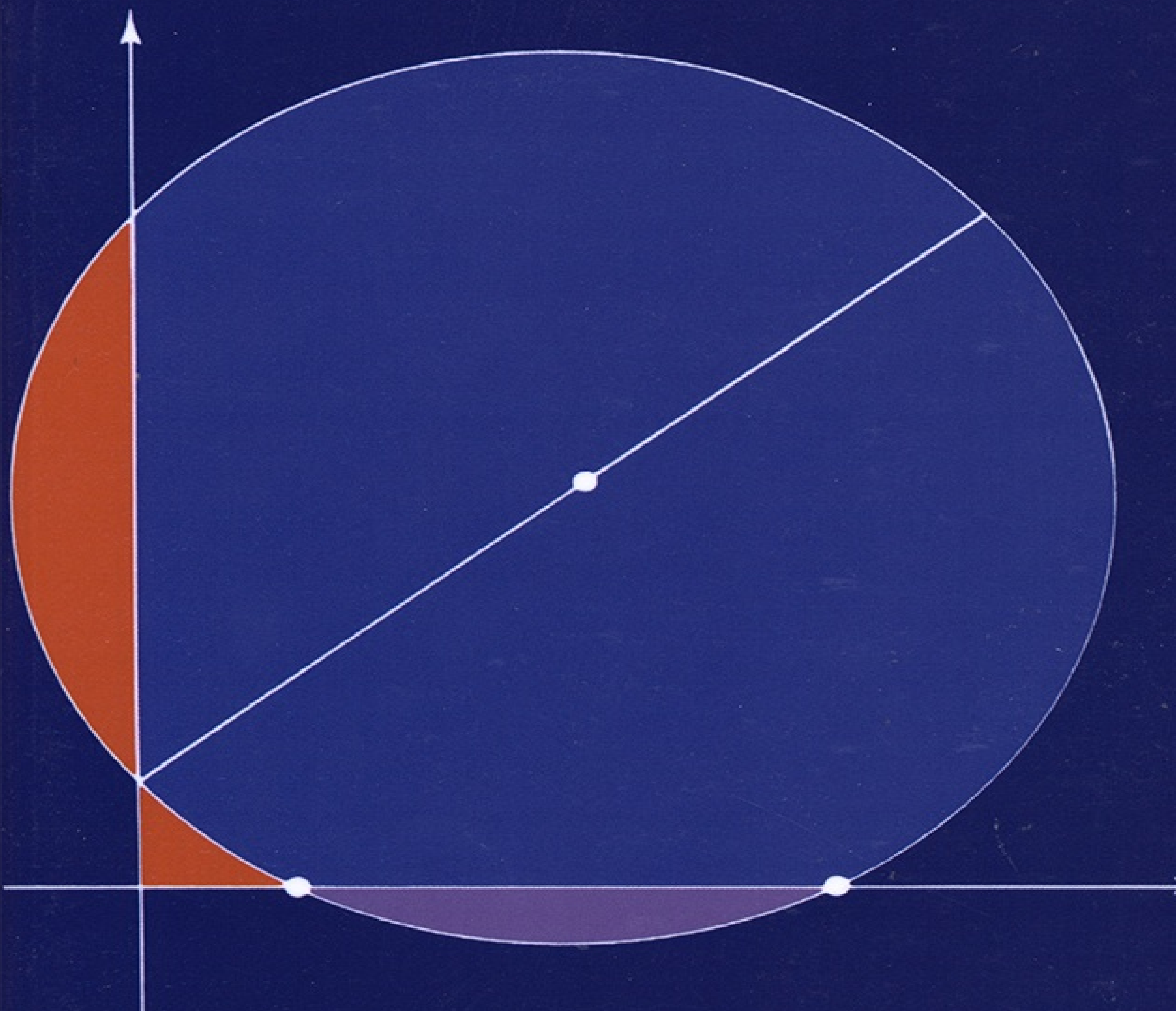
Benjamin Bold

Famous Problems of GEOMETRY and How to Solve Them



Famous Problems of Geometry and How to Solve Them

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To My Wife, CLAIRE,

Whose Assistance and Encouragement
Made this Book Possible

Contents

Foreword

IN JUNE OF 1963 a symposium on Mathematics and the Social Sciences was sponsored by th American Academy of Political and Social Sciences. One of the contributions was by Osc Morgenstern, who, together with John Von Neumann, had written the book "The Theory of Games an Economic Behavior." This book stimulated the application of mathematics to the solution of problen in economics, and led to the development of the mathematical "Theory of Games." Dr. Morgenstern contribution to the Symposium was called the "Limits of the Uses of Mathematics in Economics." shall quote the first paragraph of this article.

> Although some of the profoundest insights the human mind has achieved are best stated in negative form, it is exceedingly dangerous to discuss limits in a categorical manner. Such insights are that there can be no perpetuum mobile, that the speed of light cannot be exceeded, that the circle cannot be squared using ruler and compasses only, that similarly an angle cannot be trisected, and so on. Each one of these statements is the culmination of great intellectual effort. All are based on centuries of work and either on massive empirical evidence or on the development of new mathematics or both. Though stated negatively, these and other discoveries are positive achievements and great contributions to human knowledge. All involve mathematical reasoning; some are, indeed, in the field of pure mathematics, which abounds in statements of prohibitions and impossibilities."

The above quotation states clearly and forcefully the purpose of this book. Why does mathemati abound in "statements of prohibitions and impossibilities?" Why are the solutions of such problems "squaring the circle" and "trisecting an angle" considered to be "profound insights" and "gre contributions to human knowledge?" Why were centuries of "great intellectual effort" required solve such seemingly simple problems? And, finally, what new mathematics had to be developed resolve these problems? I hope you will find the answers to these questions as you read this book.

The outstanding achievement of the Greek mathematicians was the development of a postulation system. Despite the flaws and defects of euclidean geometry as conceived by the ancient Greeks, the work serves as a model that is followed even up to the present day.

In a postulational system one starts with a set of unproved statements (postulates) and deduces (b means of logic) other statements (theorems). Two of the postulates of euclidean plane geometry are:

1) given any two distinct points, there exists a unique line through the two points.
2) given a point and a length, a circle can be constructed with the given point as center and th given length as radius.

These two postulates form the basis for euclidean constructions (constructions using only a unmarked straight edge and compasses). With these two instruments the Greek mathematicians we able to perform many constructions; but they also were unsuccessful in many instances. Thus, the were able to bisect any given angle, but were unable to trisect a general angle. They were able construct a square equal in area to twice a given square, but were unable to "duplicate a cube." The were able to construct a square equal in area to a given polygon, but were unable to "square a circle They were able to construct regular polygons of 3, 4, 5, 6, 8 and 10 sides, but were unable to constru regular polygons of 7 or 9 sides. Before the end of the 19th century, mathematicians had supplie answers to all of these problems of antiquity. The purpose of this book is to show how these problen were eventually solved.

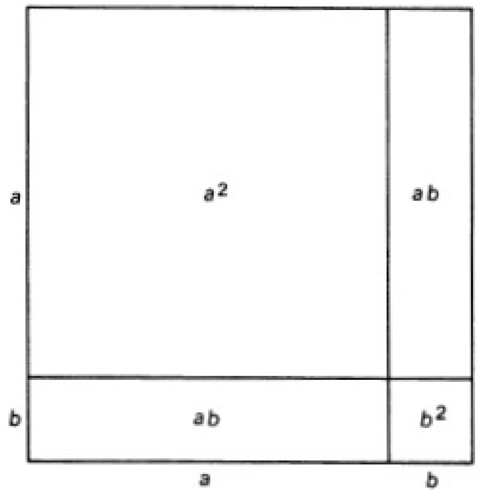
Why were the Greek mathematicians unable to solve these problems? Why was there a lapse about two thousand years before solutions to these problems were found?

~~The mathematical efforts of the Greeks were along geometric lines. The concentration on geometr~~ and the resulting neglect of algebra, was due to the following situation:

The Pythagorean theorem tells us that, if the length of a side of a square is one unit, the length the diagonal is $\sqrt{2}$ units. What kind of number is $\sqrt{2}$? The Greek mathematicians, up to this poi were able to express all their results in terms of integers. Fractions, or rational numbers, are order pairs of integers—i.e., numbers of the form a/b, where a and b are integers, $b \neq 0$. No matter how th tried, the Greeks were unable to express $\sqrt{2}$ in terms of integers. As you already know, it can proven that $\sqrt{2}$ is irrational, and it was not until the 19th century that a satisfactory theory irrationals was developed.

Because of the lack of such a theory, the course of Greek mathematics took a geometric turn. Thu when the Greeks wished to expand $(a + b)^2$, they proceeded geometrically as follows:

$$(a + b)^2 = a^2 + 2ab + b^2$$



As we shall show later the solution of the construction problems involves well developed algebra techniques, and in fact, it was not until algebra and analytic geometry were developed in the 17 century by Vieta, Descartes, Fermat, and others, that procedures were obtained that could be used in successful attack on the construction problems.

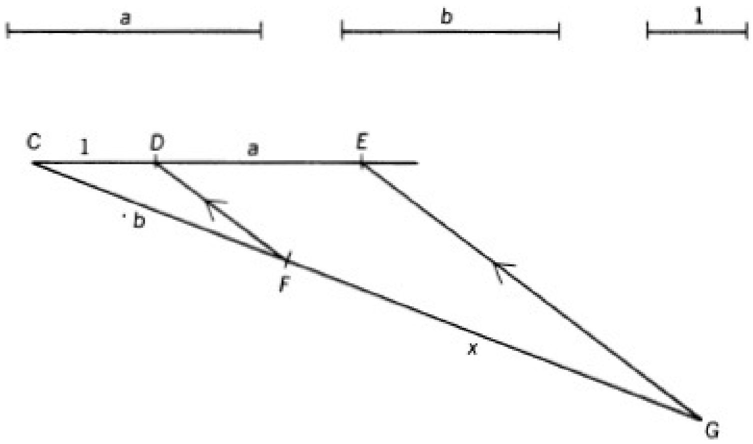
Famous Problems of Geometry and How to Solve Them

Achievement of the Ancient Greeks

USING GEOMETRIC THEOREMS, the Greek mathematicians were able to construct any desired geometr element that could be derived by a finite number of rational operations and extractions of real squa roots from the given elements. To illustrate: Suppose we are given the elements a, b, and the ur element. The Greeks could construct $a + b$, $a - b$, $a \cdot b$, a/b, a^2, and $\sqrt{a}$.

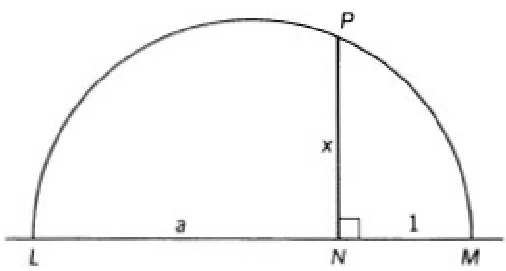
PROBLEM SET I–A

Construct $a + b$ and $a - b$, using the given line segments. The following diagram shows how construct ab. If EG is constructed parallel to DF, then $x = ab$.



PROBLEM SET I–B

1. Prove that $x = ab$.
2. Using a similar procedure construct a^2; a/b; a^2/b The diagram below shows how to construct $\sqrt{a}$.



A semicircle is constructed on LM as a diameter. NP is perpendicular to LM (P is the intersection c the perpendicular and the semicircle). Then $x = \sqrt{a}$.

PROBLEM SET I–C

1. Using the diagram above, prove $x = \sqrt{a}$.
2. Using a similar procedure construct

$$\sqrt{ab};\quad \sqrt[4]{a};\quad \sqrt[8]{a}$$

3. Using the Pythagorean theorem construct line segments equal to

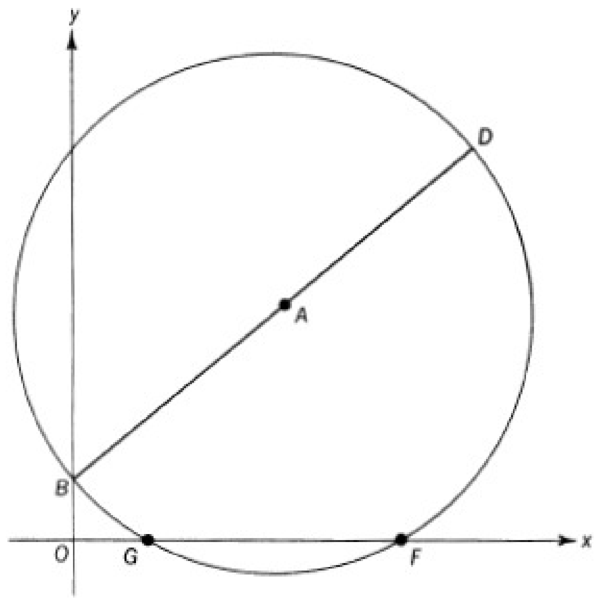
$$\sqrt{2};\quad \sqrt{3};\quad \sqrt{5};\quad \sqrt{17}$$

Using these constructions, the Greeks were able to construct the roots of a linear or quadrat equation if the numbers representing the coefficients were the lengths of given line segments.

PROBLEM SET I–D

Construct the root of $ax + b = c$, where a, b, c are given line segments.

To construct the roots of the quadratic equation $x^2 - ax + b = 0$ ($a^2 > 4b$), one can proceed follows: Construct a circle whose diameter BD joins the points B(0, 1) and D(a, b). Then the absciss of G and F (the points where the circle intersects the X-axis) will be the roots of the quadrat equation.



PROBLEM SET I–E

1. Why do we use the restriction $a^2 > 4b$?
2. Prove that the abscissas of G and F are the roots of $x^2 - ax + b = 0$. Hint: Show that the equation the circle is

$$\left(x - \frac{a}{2}\right)^2 + \left(y - \frac{b+1}{2}\right)^2 = \frac{a^2}{4} + \frac{(b-1)^2}{4}$$

As pointed out in the Introduction, the Greeks (using the basic constructions outlined in th sec~~tion) achieved considerable success in construction problems. Yet they left a number of un~~solve problems for future generations of mathematicians to struggle with. The remainder of this book w be devoted to a brief history of attempts to solve these problems and their final solution in t nineteenth century.

PROBLEM SET I–F

Construct the positive root of $x^2 + x - 1 = 0$, given a unit length.

CHAPTER II

An Analytic Criterion for Constructibility

TO ANSWER THE QUESTION "Which constructions are possible with unmarked straight edge ar compasses?" it is necessary to establish an analytic criterion for constructibility. Every constructic problem presents certain given elements a, b, c . . . and requires us to find certain other elements x, z . . . . The conditions of the problem enable us to set up one or more equations whose coefficients wi be numbers representing the given elements a, b, c . . . . The solutions of the equations will permit u to express the unknown elements in terms of the given elements. To take a simple example, suppo we wish to construct a square equal in area to twice a given square whose side is a. We express th problem analytically by the equation

$$x^2 = 2a^2$$

Other problems, of course, will lead to equations of higher degree.

We have already seen how the roots of a linear or quadratic equation can be constructed. We sha now investigate the possibility of constructing the roots of an equation of degree greater than 2, ar we shall be especially interested in the roots of cubic equations.

First, we know that it is possible to construct any geometric element if that element can be derive from the given elements by a finite number of rational operations (addition, subtractio multiplication and division), and the extraction of real square roots. Now let us consider the conver situation. If a construction is possible, what is the relation between the required elements and th given elements? It is easy to see that only those constructions are possible for which the numbe which define the desired elements can be obtained from the given elements by a finite number rational operations and the extractions of real square roots.

Any construction consists of a sequence of steps, and each step is one of the following:

1 drawing a straight line between two points.
2 constructing a circle with a given center and given radius.
3 finding the points of intersection of two straight lines, two circles, or a straight line and a circl

Let us assume that we are given a set of coordinate axes and a unit length and that all the give elements can be represented by rational numbers. We know that the sum, difference, product, ar quotient (division by 0 is always excluded) of two rational numbers is a rational number. The ration numbers are said to form a closed set with respect to the four fundamental operations. Any set numbers closed with respect to these four operations is called a field. Let us represent the field rational numbers by F_0.

If we are given the coordinates of two points

$$P_1(x_1,y_1) \text{ and } P_2(x_2,y_2)$$

then the equation of the line thru P_1 P_2 is

$$(y_2 - y_1)x + (x_1 - x_2)y + (x_2y_1 - x_1y_2) = 0$$

or

$$ax + by + c = 0$$

where

$$a = y_2 - y_1;\ b = x_1 - x_2;\ \text{and } c = x_2 y_1 - x_1 y_2$$

Note that x_1, x_2, y_1, and y_2 are rational numbers by definition. Then a, b, and c are also ration numbers.

The equation of a circle, whose center is (h, k) and whose radius is r, is

$$x^2 + y^2 - 2hx - 2ky + h^2 + k^2 - r^2 = 0$$

or

$$x^2 + y^2 + dx + ey + f = 0$$

where $d = -2h$, $e = -2k$ and $f = h^2 + k^2 - r^2$. Again, d, e, and f are rational numbers. Finding th coordinates of the point of intersection of two straight lines involves only rational operatio performed on the coefficients of the variables, whereas finding the coordinates of the intersection of straight line and a circle, or of two circles, would involve, in addition to the rational operations, on the extraction of square roots. To summarize, we can state that a proposed construction with unmarke straight edge and compasses is possible *if and only if* the numbers which define the desired elemen can be derived from the given elements by a finite number of rational operations and the extractio of square roots.

PROBLEM SET II–A

Find the coordinates of the points of intersection of

(a) $ax + by + c = 0$ and $a'x + b'y + c' = 0$
(b) $ax + by + c = 0$ and $x^2 + y^2 + dx + ey + f = 0$
(c) $x^2 + y^2 + dx + ey + f = 0$ and

$$x^2 + y^2 + d'x + e'y + f' = 0$$

All rational numbers, i.e., all numbers in F_0, can be constructed if we are given a unit lengt Furthermore, if k is a fixed number in F_0 we can construct $\sqrt{k}$ and $a + b\sqrt{k}$, where a and b are ar numbers in F_0. If $\sqrt{k}$ is not in F_0, then we can prove that all numbers of the form $a + b\sqrt{k}$ $(a,b,k$ $F_0)$ form a new field F_1, which has F_0 as a subfield. For example, let $k = 3$, then $a + b\sqrt{3}$ form a fie F_1 which contains all rational numbers as a subfield, namely all those numbers for which $b = 0$.

To prove that all numbers $a + b\sqrt{k}$ $(a, b, k$ in F_0, $\sqrt{k}$ not in $F_0)$ form a field we observe that

1. $(a + b\sqrt{k}) + (c + d\sqrt{k}) = (a + c) + (b + d)\sqrt{k}$
$\qquad = e + f\sqrt{k}$
2. $(a + b\sqrt{k}) - (c + d\sqrt{k}) = (a - c) + (b - d)\sqrt{k}$
$\qquad = m + n\sqrt{k}$
3. $(a + b\sqrt{k}) \times (c + d\sqrt{k}) = (ac + bdk) + (ad + bc)\sqrt{k}$
$\qquad = r + s\sqrt{k}$

4. $\dfrac{a+b\sqrt{k}}{c+d\sqrt{k}}\cdot\dfrac{c-d\sqrt{k}}{c-d\sqrt{k}}=\dfrac{(ac-bdk)+(bc-ad)\sqrt{k}}{c^2-d^2k}$

$$=\frac{ac-bdk}{c^2-d^2k}+\frac{(bc-ad)\sqrt{k}}{c^2-d^2k}$$
$$=u+v\sqrt{k}$$

Since a, b, c, d, k are numbers in the field F_0, the sum, product, difference, and quotient of any tw of these rational numbers is a rational number. Therefore e, f, m, n, r, s, u, and v are rational number which proves that all numbers of the form $a+b\sqrt{k}$ form a field F_1, which contains F_0 as a subfiel There is one detail which requires clarification. When $(a+b\sqrt{k})$ was divided by $(c+d\sqrt{k})$ v obtained $u+v\sqrt{k}$

$$u=\frac{ac-bdk}{c^2-d^2k}$$

and

$$v=\frac{bc-ad}{c^2-d^2k}$$

Thus u and v are rational numbers if and only if

$$c^2-d^2k\neq 0$$

We have assumed that $\sqrt{k}$ is not in F_0. Therefore, $k\neq 0$. Also when we divide we assume the divis is not 0. Therefore c and d are not both 0, although either c or d may be 0. We now have to show th $c^2-d^2k\neq 0$. If $c^2-d^2k=0$, then $c^2=d^2k$, $k=c^2/d^2$ and $\sqrt{k}=\pm c/d$. Thus $\sqrt{k}$ would be in contrary to the hypothesis. This completes the proof that all numbers of the form $a+b\sqrt{k}$ form field, F_1.

Next we can construct all numbers of the form $a_1+b_1\sqrt{k_1}$ where a_1 and b_1 are any numbers in k_1 is a fixed number in F_1, and the $\sqrt{k_1}$ is not in F_1. Such a number would be $\sqrt{5}+\sqrt{2}\sqrt[4]{3}$, wher $a_1=\sqrt{5}$, $b_1=\sqrt{2}$, $k_1=\sqrt{3}$ are in F_1 and $\sqrt{k_1}=\sqrt[4]{3}$ is not in F_1. Another example would b $5+2\sqrt[4]{3}$. Again we can prove by a process completely analogous to that used above that all numbe of the form $a_1+b_1\sqrt{k_1}$ form a field F_2, which has F_1 as a subfield (which contains numbers of t form $a_1+b_1\sqrt{k_1}$, where $b_1=0$).

The process can be continued indefinitely until we reach a field F_n, n, a positive integer. We no have a sequence of fields $F_0, F_1, F_2, \ldots F_n$ with the following properties:

1. F_0 is the field of rational numbers.

F_1 is the field obtained by adjoining $\sqrt{k}$ to F_0, where k is a fixed number in F_0 such that $\sqrt{k}$ not in F_0; F_1 contains all numbers of the form $a+b\sqrt{k}$, a, b, k in F_0 $\sqrt{k}$ not in F_0.

F_2 is the field obtained by adjoining $\sqrt{k_1}$ to F_1, where k_1 is a fixed number in F_1 such that $\sqrt{k_1}$ not in F_1. The field F_2 contains all numbers of the form $a_1+b_1\sqrt{k_1}; a_1, b_1, k_1$ in F_1, $\sqrt{k_1}$ not in F_1.

F_n is the field obtained by adjoining $\sqrt{k_{n-1}}$ to F_{n-1}, where k_{n-1} is a fixed number in F_{n-1} such tha $\sqrt{k_{n-1}}$ is not in F_{n-1}. F_n contains all numbers of the form

$$a_{n-1}+b_{n-1}\sqrt{k_{n-1}};\ a_{n-1}, b_{n-1}, k_{n-1}\quad\text{in}\quad F_{n-1}$$

but $\sqrt{k_{n-1}}$ is not in F_{n-1}. A number in F_n may be exceedingly complicated, consisting as it does of square roots, one over the other. An example of a number in F_4 would be

$$\sqrt{2} - \sqrt{\sqrt{3 + \sqrt{7 + \sqrt{5}}}}$$

To obtain the number

$$\sqrt{\sqrt{3 + \sqrt{7 + \sqrt{5}}}}$$

we can start with $k_0 = 5$, then $k_1 = 7 + \sqrt{k_0}$, $k_2 = 3 + \sqrt{k_1}$ $k_3 = \sqrt{k_2}$, and $k_4 = \sqrt{k_3}$.. This number i therefore, in F_4.

2. F_0 is a subfield of F_1; F_1 is a subfield of F_2; F_{n-1} is a subfield of F_n.

3. Every number in $F_0, F_1, F_2, \ldots, F_n$ is constructible, since a number in any one of these fields ca be obtained from the unit element by a finite number of rational operations and extractions of squa roots.

4. Conversely any constructible number can be found in one of the fields $F_0, F_1, F_2, \ldots, F_n$, sin we have already shown that only those constructions are possible for which the numbers which defi the desired elements can be obtained from the given elements (which we assumed to be represented b rational numbers) by a finite number of rational operations and extractions of square roots.

As a summary we can state the following theorem—All the numbers in the fields $F_0, F_1, \ldots, F_n$ a constructible, and conversely, any constructible number must be in one of the fields $F_0, F_1, \ldots, F$ Thus, $\sqrt{3 + \sqrt[8]{2}}$ is constructible, since $\sqrt{3 + \sqrt[8]{2}}$ is in F_4; whereas $\sqrt{3 + \sqrt[6]{2}}$ is not constructibl since $\sqrt{3 + \sqrt[6]{2}}$ is not in any one of the fields $F_0, F_1, \ldots, F_n$.

PROBLEM SET II–B

1. Show that $7/(5 - \sqrt{2})$ is in F_1 by expressing the number in the form $a + b\sqrt{k}$, where a, b, k are F_0.
2. Show that $5/(2 - \sqrt[4]{3})$ is in F_2 by expressing the number in the form $a_1 + b_1\sqrt{k_1}$, where a_1, b_1, are in F_1. Hint: To rationalize the denominator, use the identity

$$s^4 - t^4 = (s - t)(s + t)(s^2 + t^2),\ s = 2,\ t = \sqrt[4]{3}$$

We are now in a position to determine when the roots of a cubic equation are constructible, b proving the following theorem: If a cubic equation with rational coefficients has no rational root, th none of its roots is constructible. Any cubic equation with rational coefficients is said to be reducib in the field of rational numbers if it has at least one rational root. If the equation has no rational ro it is said to be irreducible in the field of rational numbers. Thus, we wish to show that no root of irreducible cubic equation can be constructed. We can represent the cubic equation by

$$x^3 + Px^2 + qx + r = 0$$

where p, q, r are in F_0. By hypothesis, the equation is irreducible, and consequently has no ration roots. Let us assume that one of the roots, x_1, is constructible. Then x_1 is a number in some field F where n is an integer > 0. x_1 cannot belong to F_0, since the equation is irreducible. Let m be t smallest integer for which x_1 belongs to F_m, i.e., x does not belong to F_{m-1} ($m > 1$). If the equation h any other constructible root, we assume that that root belongs to F_r, where $r \geq m$. Then x_1 is of t form

$$a_{m-1} + b_{m-1}\sqrt{k_{m-1}}$$

To simplify our notation, let

$$a = a_{m-1},\ b = b_{m-1},\ k = k_{m-1}$$

In other words, a, b, k belong to F_{m-1}, and $x_1 = a + b\sqrt{k}$ belongs to F_m but not to F_{m-1}.

We now show that if $a + b\sqrt{k}$ is a root of the cubic equation

$$x^3 + Px^2 + qx + r = 0$$

then $a - b\sqrt{k}$ must also be a root. If $a + b\sqrt{k}$ is a root of the equation, then

$$(a + b\sqrt{k})^3 + p(a + b\sqrt{k})^2 + q(a + b\sqrt{k}) + r = 0$$

Simplifying, we obtain

$$a^3 + 3a^2b\sqrt{k} + 3ab^2k + b^3k\sqrt{k} + pa^2 + 2pab\sqrt{k} + pb^2k + qa + qb\sqrt{k} + r = 0$$

or $s + t\sqrt{k} = 0$, where

$$s = a^3 + 3ab^2k + pa^2 + pb^2k + qa + r \quad \text{and} \quad t = 3a^2b + b^3k + 2pab + qb$$

Thus, either $\sqrt{k} = -s/t$ or $s = 0$ and $t = 0$. However, t and s are numbers in F_{m-1}. Therefore, i $\sqrt{k} = -s/t$, $\sqrt{k}$ would also be a number in F_{m-1}, contrary to the hypothesis that m is the least integ for which x_1 belongs to F_m. Therefore, $s = 0$ and $t = 0$.

If we now substitute $a - b\sqrt{k}$ for x in the polynomial

$$x^3 + px^2 + qx + r = 0$$

we obtain the expression $s' + t'\sqrt{k}$, where $s' = a^3 + 3ab^2k + pa^2 + pb^2k + qa + r$ and $t' = -(3a^2b + b$ $+ 2pab + qb)$. Thus $s' = s = 0$, and $t' = -t = 0$. Therefore, $s' + t'\sqrt{k} = 0$ and $a - b\sqrt{k}$ is a root of t cubic equation.

We now know that $x_1 = a + b\sqrt{k}$ and $x_2 = a - b\sqrt{k}$. To find the third root x_3, we observe that t cubic equation can be written in the form

$$(x - x_1)(x - x_2)(x - x_3) = 0$$
$$x^3 - (x_1 + x_2 + x_3)x^2 + (x_1x_2 + x_1x_3 + x_2x_3)x - x_1x_2x_3 = 0.$$

Therefore

$$x_1 + x_2 + x_3 = -p$$

or, since $x_1 + x_2 = 2a$, $x_3 = -2a - p$, which means that one of the roots, x_3, is in the field F_{m-1}, contra to the hypothesis that m is the least integer such that F_m contains a root of the cubic equation.

We can, therefore, conclude that if a cubic equation with rational coefficients has a constructib root, it also has a rational root. If we represent the rational root by x_r, we can write the cubic equatio as

$$(x - x_r)(x^2 + p_1x + q_1) = 0$$

Consequently, the other two roots are roots of a quadratic equation and are also constructibl Conversely, if a cubic equation with rational coefficients has a rational root, we can write the equatio in the form

$$(x - x_r)(x^2 + p_1 x + q_1 = 0$$

where p_1 and q_1 are rational. Thus the roots of this equation are constructible. To summarize, we ca state the following theorem:

The roots of a cubic equation with rational coefficients are constructible if and only if the equatio has a rational root. If the equation is irreducible, then none of its roots can be constructed with straig edge and compasses.

This property of cubic equations is basic in solving most of the construction problems the Greel were unable to dispose of. As later chapters will show, this theorem can be used to solve the followin problems—duplicating a cube, trisecting an arbitrary angle, constructing a regular polygon of 7 sid and 9 sides. The problem of squaring a circle requires results of a different nature, as the developme will show. Before we undertake the assault on these problems, there will be a brief interlude o Complex Numbers.

Finally, it is interesting to observe that the theorem on cubic equations is a special case of a mo general theorem whose proof is beyond the scope of this book. We first define the term *irreducible* a more general context: Let $p(x) = a_0x^n + a_1x^{n-1} + . . . + a_n = 0$ be a polynomial equation of degree where n represents a positive integer and a_i represents a rational number. Then $p(x) = 0$ is reducible the field of rational numbers if $p(x)$ can be factored into polynomials of *lower* degree with coefficien in the field of rational numbers. If $p(x)$ cannot be factored in that manner then $p(x)$ is said to b irreducible. It can now be proven that a geometric element is constructible if, and only if, the numb representing the element is the root of an irreducible polynomial equation (with rational coefficient of degree 2^k, where k represents a non-negative integer.

PROBLEM SET II–C

1. Show that $\sqrt{3 + \sqrt[4]{2}}$ is the root of an irreducible equation of degree 2^3.
2. Construct $\sqrt{3 + \sqrt[4]{2}}$.
3. Which one of the cubic equations has a rational root
   (a) $x^3 - 1 = 0$
   (b) $x^3 - 2 = 0$
4. Find the three roots of the reducible cubic equation.
5. Given a cubic equation $x^3 + ax^2 + bx + c = 0$, where a, b, and c are rational numbers, and who roots r_1, r_2, and r_3 are positive real numbers; if r_1 is rational, show that all the roots of th equation are constructible by showing that the roots are numbers in F_0 or F_1. As an illustratio construct the roots of $x^3 - 7x^2 + 14x - 6 = 0$ given a unit length.
6. If $2 + \sqrt{3}$ is a root of the cubic equation $x^3 + ax^2 + bx + c = 0$, where a, b, and c are ration numbers, show that one of the roots of the equation is rational.

Complex Numbers

YOU MAY WONDER what complex numbers have to do with constructing geometric lines or figures. B as you progress you will observe that complex numbers are most useful in both algebraic ar geometric problems involving real numbers.

You will recall that finding the diagonal of a square of unit side led to the equation $x^2 = 2$, and th the solution of this equation proved to be a real obstacle to the progress of Greek mathematics. Tl Greeks could obtain the rational numbers by forming ordered pairs of integers (a/b, $b \neq 0$). Howeve there was no such simple procedure for proceeding from the rationals to the reals (including irration numbers).

Another roadblock in the history of mathematics resulted from the equation $x^2 + 1 = 0$. You shoul be able to prove that there is no real number x whose square is -1. (The square of a positive negative number is positive, and the square of 0 is 0.) There are two alternatives: either we mu assume that the simple equation $x^2 + 1 = 0$ has no solution, or we must extend the real number syste in such a way that the new number field will contain a solution of $x^2 + 1 = 0$. The latter alternative i of course, the more desirable one, and the new number field can be obtained easily by following procedure similar to that used in forming rational numbers from the integers, i.e., we form a comple number by using an ordered pair of real numbers with suitable definitions for equality, addition, ar multiplication. We can represent a complex number in the form $a + bi$ ($i = \sqrt{-1}$). The definitions f equality, addition, and multiplication will be as follows:

1 $a + bi = c + di$ if and only if $a = c$ and $b = d$

2 $(a + bi) + (c + di) = (a + c) + (b + d)\, i$

3 $(a + bi)\cdot(c + di) = (ac - bd) + (ad + bc)\, i$

These are the definitions you would expect if we consider i to behave as any other variable such as with the restriction that $i^2 = -1$.

PROBLEM SET III–A

For which values of a and b will $a + bi$ be a solution of $x^2 + 1 = 0$.

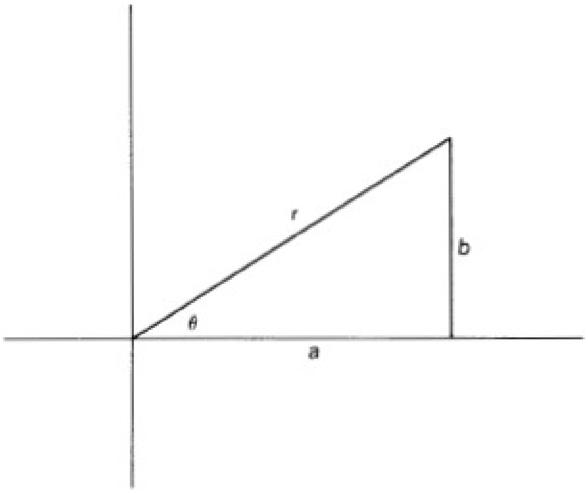
It is not necessary to use i in representing a complex number. We can merely write (a, b), where and b are real numbers.

PROBLEM SET III–B

1. Write the definitions for equality, addition, and multiplication, using the form (a, b) to represent complex number.
2. If we write the additive identity for complex numbers as $(0, 0)$, and the multiplicative identity

(1, 0), find the solution of $x^2 + (1, 0) = (0, 0)$ and check your answers.

You are no doubt familiar with the fact that polar coordinates (r, θ) as well as rectangul coordinates (a, b) can be used to represent a complex number. Then, in polar form, the comple number $a + bi$, or (a, b) can be represented as $r(\cos\theta + i\sin\theta)$.



The polar form of complex numbers enables us to multiply two complex numbers very easily.

$$r_1(\cos\theta_1 + i\sin\theta_1)\cdot r_2(\cos\theta_2 + i\sin\theta_2) = r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)]$$

i.e., when we multiply two complex numbers we multiply the moduli and add the amplitudes.

PROBLEM SET III–C

1. Prove the last statement using the trigonometric identities

$$\sin(x + y) = \sin x\cos y + \cos x\sin y$$
$$\cos(x + y) = \cos x\cos y - \sin x\sin y$$

2. Prove that

$$\frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} = \frac{r_1}{r_2}[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)]$$

If we let $r_1 = r_2 = 1$ and $\theta_1 = \theta_2$ we obtain

$$(\cos\theta + i\sin\theta)^2 = \cos 2\theta + i\sin 2\theta$$

The last equation is an illustration of De Moivre’s theorem:

$$(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$$

For m a positive integer, the theorem can be proven by mathematical induction. The theorem certain holds for $m = 1$. If we assume it holds for m, then $(\cos\theta + i\sin\theta)^m = \cos m\theta + i\sin m\theta$. Therefore (c $\theta + i\sin\theta)^{m+1} = (\cos\theta + i\sin\theta)(\cos m\theta + i\sin m\theta) = \cos\theta\cos m\theta - \sin\theta\sin m\theta + i(\sin\theta\cos m\theta$ $\cos\theta\sin m\theta) = \cos(m+1)\theta + i\sin(m+1)\theta$

Thus, the theorem is shown to hold for all positive integers. However, De Moivre’s theorem can b established for all real or complex values of m. Let us use De Moivre's theorem to obtain a usef formula for *real numbers.*

If x is a real number, then $(\cos x + i \sin x)^2 = \cos 2x + i \sin 2x$. But $(\cos x + i \sin x)^2 = (\cos^2 x - \sin^2 x) + 2i \sin x \cos x$, and $\cos 2x + i \sin 2x = (\cos^2 x - \sin^2 x) + 2i \sin x \cos x$. Therefore, $\cos 2x = \cos^2 x - \sin^2 x$, and $\sin 2x = 2 \sin x \cos x$. Observe that we were able to obtain formulas involving only real numbers by equating the real and imaginary parts of two equal complex numbers.

PROBLEM SET III–D

Using the above procedure, prove

1. $\cos 3x = 4 \cos^3 x - 3 \cos x$
2. $\sin 3x = 3 \sin x - 4 \sin^3 x$

Hint: After equating real and imaginary parts of the complex numbers, use the relation $\sin^2 x + \cos^2 x = 1$.

Note that we shall use the first formula of Problem Set III–D when we discuss the problem of trisecting an angle.

De Moivre's theorem enables us to find the roots of unity in a very simple manner. (We shall use the roots of unity when we discuss the problem of constructing regular polygons). To find the two square roots of unity we may proceed algebraically as follows:

$$x^2 = 1 \text{ and } x = +1 \text{ or } x = -1$$

Using De Moivre's theorem we would proceed as follows: Let $R = \cos\theta + i \sin\theta$, where R is a square root of unity. Then $R^2 = \cos 2\theta + i \sin 2\theta = 1$, since R is a square root of unity. Therefore $2\theta = 2k\pi$, where k is any integer. However, only 2 values of k will give distinct values of R. Thus

$$R_1 = \cos \pi + i \sin \pi = -1 \ (k = 1)$$
$$R_2 = \cos 2\pi + i \sin 2\pi = 1 \ (k = 2)$$

No advantage is evident in using De Moivre's theorem in the above example. However, should we wish to find the 17 seventeenth roots of unity, then the power of De Moivre's theorem would soon become clear.

Let us try finding the three cube roots of unity. Algebraically we use the equation

$$x^3 - 1 = 0$$
$$(x-1)(x^2 + x + 1) = 0$$
$$x_1 = 1,\ x_2 = \frac{-1+\sqrt{-3}}{2},\ x_3 = \frac{-1-\sqrt{-3}}{2}$$

Observe that two of the roots are imaginary numbers.

By De Moivre's theorem we have

$$R = \cos\theta + i\sin\theta$$
$$R^3 = \cos 3\theta + i\sin 3\theta = 1$$
$$3\theta = 2k\pi,\ k = 1, 2, 3$$
$$\theta = \frac{2k\pi}{3},\ k = 1, 2, 3$$
$$R_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$$
$$R_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$$
$$R_3 = \cos\frac{6\pi}{3} + i\sin\frac{6\pi}{3} = 1$$

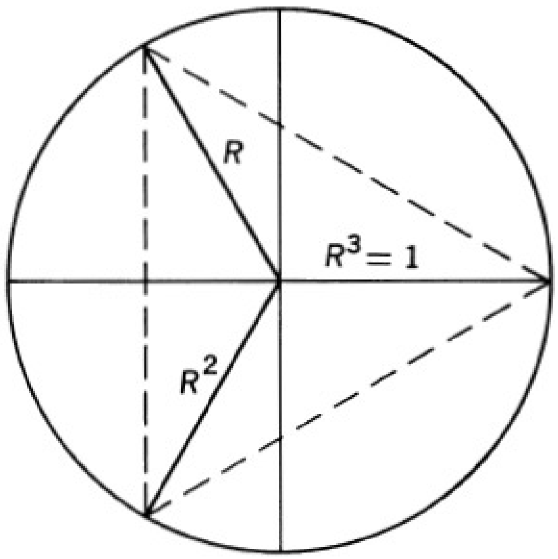
PROBLEM SET III–E

Show that $R_1 = x_2$, $R_2 = x_3$, $R_3 = x_1$.

Observe that (again using De Moivre's theorem)

$$R_2 = R_1^2,\ R_3 = R_1^3$$

Therefore, we can write the three cube roots of unity as R, R^2 and R^3. Also observe that the cube roo of unity, when plotted, divide the unit circle into three equal parts and that an equilateral triangle ca be formed by joining the points of division.



PROBLEM SET III–F

1. If we let $x_2 = (-1 + \sqrt{-3})/2 = \omega$, prove that $x_3 = \omega^2$ and $x_1 = \omega^3$. Does a similar relation hold f the two square roots of unity?

Let us now generalize the procedure and find expressions for the n roots of the equation $x^n = 1$ or $- 1 = 0$. By using the formula for the sum of geometric series or simply by multiplying, one can sho that

$$\frac{x^n - 1}{x - 1} = x^{n-1} + x^{n-2} + \cdots + x + 1 \quad (x \neq 1)$$

If we let $R(\neq 1)$ be an n nth root of unity, then all the roots can be expressed as $R, R^2, R^3, \ldots, R^{n-1}$, … = 1, since an nth root of unity can be written in the form

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},\ k = 1, 2, \cdots, n$$

Thus for $k = 1$,

$$R_1 = \frac{\cos 2\pi}{n} + i\sin\frac{2\pi}{n}$$

for $k = 2$,

$$R_2 = \frac{\cos 4\pi}{n} + i\sin\frac{4\pi}{n} = R_1^2$$

for $k = n$,

$$R_n = \cos 2\pi + i\sin 2\pi = R^n = 1$$

The n nth roots of unity, when plotted in the complex plane, divide the unit circle into n equal arc and joining the arcs will result in a regular n-gon. Since the roots of the equation $x^{n-1} + n^{n-2} + \ldots +$ $+ 1 = 0$ are the complex nth roots of unity and together with the root $x = 1$, divide the circle into equ parts, the equation is called the "cyclotomic" (circle dividing) equation. The numbers $R, R^2, \ldots, R^n$ 1 form a multiplicative group since they satisfy the following four conditions:

1 closure $R^a \cdot R^b = R^{a+b} = R^c$, where a, b, c are integers $\leq n$
2 associativity $R^a(R^b \cdot R^c) = (R^a \cdot R^b) \cdot R^c = R^{a+b+c}$
3 identity element is R^n since $R^a \cdot R^n = R^a$
4 inverse element for R^a is R^{n-a}

Also observe that the inverse of R ($\cos 2\pi/n + i \sin 2\pi/n$) is R^{-1} which can be written in the form c $(2\pi/n) - i\sin(2\pi/n)$.

PROBLEM SET III–G

1. Express the seven 7th roots of unity ($R, R^2, \ldots, R^7 = 1$) in polar form.
2. The seven 7th roots of unity form a group. Show that
   (a) $R^3 \cdot R^6$ is a member of the group.
   (b) The inverse of R^5 is a member of the group.
   (c) $R^2 + (1/R^2) = 2\cos 4\pi/7$
3. By using the formula $S = (ar^n - a)/(r - 1)$, show that

$$1 + R + R^2 + \cdots + R^{n-1} = \frac{R^n - 1}{R - 1}$$

The Delian Problem

THE PROBLEM OF CONSTRUCTING a cube whose volume shall be twice that of a given cube is known the Delian problem. D. E. Smith in his History of Mathematics relates the following story wi reference to this problem: ". . . the Athenians appealed to the oracle at Delos to know how to stay th plague which visited their city in 430 B.C. It is said that the oracle replied that they must double in si the altar of Apollo. This altar being a cube, the problem was that of its duplication."

The problem of duplicating a cube whose edge is one unit leads to the equation $x^3 = 2$, which is a irreducible cubic equation. For if a solution of $x^3 = 2$ were rational, then we could represent th solution by a/b, a and b integers, $b \neq 0$. Let a/b be in lowest terms, i.e., a and b have no common fact greater than 1. Then $a^3 = 2b^3$ and a^3 would be an even integer. Therefore, a would be an even intege say $2n$, since the cube of an odd integer is odd. Thus $(2n)^3 = 2b^3$

$$b^3 = 4n^3$$

b^3 is even, and b is even, which contradicts the hypothesis that a and b have no common factor great than 1. Since $x^3 = 2$ is an irreducible cubic equation, its roots cannot be constructed with unmarke straight edge and compasses, and it is not possible to duplicate the cube with those instruments.

Early attempts by Hippocrates and Menaechmus showed that the problem could be solved b finding the intersections of parabolas and hyperbolas. Thus the equations $x^2 = ay$ and $y^2 = bx$ result the equation $x^3 = 2a^3$, if we let $b = 2a$.

PROBLEM SET IV–A

1. Derive the equation $x^3 = 2a^3$ from $x^2 = ay$ and $y^2 = bx$.
2. Why is this method not considered to be a "solution" of the Delian problem?
3. Show that $x^2 = ay$ and $xy = ab$, with $b = 2a$, also lead to the equation $x^3 = 2a^3$.

Diocles (second century B.C.) used the cissoid to duplicate the cube. Vieta, Descartes, Fermat, an Newton also developed methods for duplicating the cube. Newton used the limacon of Pascal for th purpose. Of course, none of the methods were restricted to the use of unmarked straight edge an compasses.

PROBLEM SET IV–B

1. Although it is not possible to duplicate a cube, the two-dimensional analogue of the problem ca be solved. Construct a square whose area is twice a given square.
2. Is it possible to construct the radius of a sphere whose surface area is twice the surface area of

unit sphere, or whose volume is twice the volume of a unit sphere?