# Differential Geometry from a Singularity Theory Viewpoint

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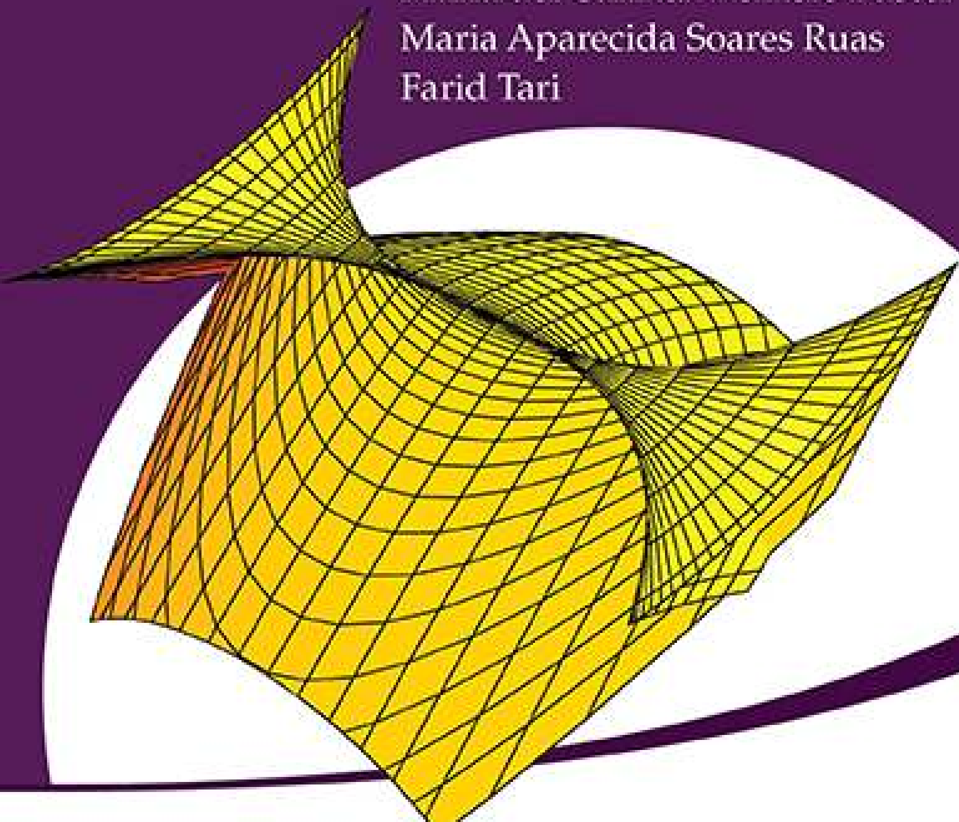

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# Preface

The geometry of surfaces is a subject that has fascinated many mathematicians and users of mathematics. This book offers a new look at this classical subject, namely from the point of view of singularity theory. Robust geometric features on a surface in the Euclidean 3-space, some of which are detectable by the naked eye, can be captured by certain types of singularities of some functions and mappings on the surface. In fact, the mappings in question come as members of some natural families of mappings on the surface. The singularities of the individual members of these families of mappings measure the contact of the surface with model objects such as lines, circles, planes and spheres.

This book gives a detailed account of the theory of contact between manifolds and its link with the theory of caustics and wavefronts. It then uses the powerful techniques of these theories to deduce geometric information about surfaces immersed in the Euclidean 3, 4 and 5-spaces as well as spacelike surfaces in the Minkowski space-time.

In Chapter 1 we argue the case for using singularity theory to study the extrinsic geometry of submanifolds of Euclidean spaces (or of other spaces). To make the book self-contained, we devote Chapter 2 to introducing basic facts about the extrinsic geometry of submanifolds of Euclidean spaces. Chapter 3 deals with singularities of smooth mappings. We state the results on finite determinacy and versal unfoldings which are fundamental in the study of the geometric families of mappings on surfaces treated in the book. Chapter 4 is about the theory of contact introduced by Mather and developed by Montaldi. In Chapter 5 we recall some basic concepts in symplectic and contact geometry and establish the link between the theory of contact and that of Lagrangian and Legendrian singularities. We apply in Chapters 6, 7 and 8 the singularity theory framework exposed in the previous chapters to the study of the extrinsic differential geometry of surfaces in the Euclidean 3, 4 and 5-spaces respectively. The codimension of the surface in the ambient space is 1, 2 or 3 and this book shows how some aspects of the geometry of the surface change with its codimension. In Chapter 9 we chose spacelike surfaces in the Minkowski space-time to illustrate how to approach the study of submanifolds in Minkowski spaces using singularity theory. Most of the results in the previous chapters are local in nature. Chapter 10 gives a flavour of global results on closed surfaces using local invariants obtained from the local study of the surfaces in the previous chapters.

The emphasis in this book is on how to apply singularity theory to the study of the extrinsic geometry of surfaces. The methods apply to any smooth submanifolds of higher dimensional Euclidean space as well as to other settings, such as affine, hyperbolic or Minkowski spaces. However, as it is shown in Chapters 6, 7 and 8, each pair $(m, n)$ with $m$ the dimension of the submanifold and $n$ of the ambient space needs to be considered separately.

This book is unapologetically biased as it focuses on research results and interests of the authors and their collaborators. We tried to remedy this by including, in the Notes of each chapter, other aspect and studies on the topics in question and as many references as we can. Omissions are inevitable, and we apologise to anyone whose work is unintentionally left out.

Currently, there is a growing and justified interest in the study of the differential geometry of singular submanifolds (such as caustics, wavefronts, images of singular mappings etc) of Euclidean or Minkowski spaces, and of submanifolds with induced (pseudo) metrics changing signature on some subsets of the submanifolds. We hope that this book can be used as a guide to anyone embarking on the study of such objects.

This book has been used (twice so far!) by the last-named author as lecture notes for a post-graduate course at the University of São Paulo, in São Carlos. We thank the following students for their thorough reading of the final draft of the book: Alex Paulo Francisco, Leandro Nery de Oliveira, Lito Edinson Bocanegra Rodríguez, Martin Barajas Sichaca, Mostafa Salarinoghabi and Patricia Tempesta. Thanks are also due to Catarina Mendes de Jesus for her help with a couple of the book's figures and to Asahi Tsuchida, Shunichi Honda and Yutaro Kabata for correcting some typos. Most of the results in Chapter 4 are due to James Montaldi. We thank him for allowing us to reproduce some of his proofs in this book.

We are also very grateful to Masatomo Takahashi for reading the final draft of the book and for his invaluable comments and corrections.

S. Izumiya, M. C. Romero Fuster, M. A. S. Ruas and F. Tari
August, 2015

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# Contents

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# Chapter 1

# The case for the singularity theory approach

The study of curves and surfaces in the Euclidean space is a fascinating and important subject in differential geometry. We highlight in this chapter how singularity theory can be used not only to recover classical results on curves and surfaces in a simpler and more elegant way but also how it reveals the rich and deep underlying concepts involved.

We start with the evolute and parallels of a plane curve. We first use classical differential geometry techniques to obtain the shape of the evolute and parallels. We then define the family of distance squared functions on the plane curve and recover from the singularities type of the members of this family geometric information about the curve itself. We outline how to use the Lagrangian and Legendrian singularity theory framework to deduce properties of the evolute that are invariant under diffeomorphisms. We proceed similarly for surfaces in the Euclidean 3-space and consider the singularities of their focal sets. We deal in the last section with the singularities of ruled and developable surfaces.

We refer to [do Carmo (1976)] for a detailed study of the differential geometry of curves and surfaces.

Throughout this book, a given map is said to be *smooth* (or $C^\infty$) if its partial derivatives of all order exist and are continuous.

The Euclidean $n$-space is the vector space $\mathbb{R}^n$ endowed with the scalar product

$$\langle \mathbf{u}, \mathbf{v} \rangle = u_1 v_1 + \cdots + u_n v_n$$

for any $\mathbf{u} = (u_1, \ldots, u_n)$ and $\mathbf{v} = (v_1, \ldots, v_n)$ in $\mathbb{R}^n$.

We also view the Euclidean $n$-space as a set of points. The vector space $\mathbb{R}^n$ comes with a standard orthogonal basis $\mathbf{e}_1 = (1, \ldots, 0)$, $\ldots$, $\mathbf{e}_n = (0, \ldots, 1)$. We choose a point $O = (0, \ldots, 0)$ to be the origin and denote by $\Sigma = (O, \mathbf{e}_1, \ldots, \mathbf{e}_n)$ the standard orthonormal coordinates system in $\mathbb{R}^n$. Then, a point $p$ in the Euclidean $n$-space is the endpoint of the vector $Op$ and its coordinates $(x_1, \dots, x_n)$ in the system $\Sigma$ are the coordinates of the vector $Op$ in the basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$. Curves, surfaces, submanifolds in $\mathbb{R}^n$ are considered as subsets of points in $\mathbb{R}^n$.

The vector product of $n-1$ vectors $\mathbf{u}_1, \dots, \mathbf{u}_{n-1}$ in $\mathbb{R}^n$, is defined by

$$\mathbf{u}_1 \times \cdots \times \mathbf{u}_{n-1} = \begin{vmatrix} \boldsymbol{e}_1 & \cdots & \boldsymbol{e}_n \\ u_1^1 & \cdots & u_n^1 \\ \vdots & \cdots & \vdots \\ u_1^{n-1} & \cdots & u_n^{n-1} \end{vmatrix},$$

where $\mathbf{u}_i = (u_1^i, \dots, u_n^i)$. By the property of the determinant, we have

$$\langle \mathbf{u}, \mathbf{u}_1 \times \cdots \times \mathbf{u}_{n-1} \rangle = \det(\mathbf{u}, \mathbf{u}_1, \dots, \mathbf{u}_{n-1}).$$

## 1.1 Plane curves

A smooth curve in the Euclidean $n$-space is a smooth map $\gamma : I \to \mathbb{R}^n$, where $I$ is an open interval of $\mathbb{R}$. The *trace* of $\gamma$, which we still denote by $\gamma$, is the set of points $\gamma(I)$ in $\mathbb{R}^n$. The curve $\gamma$ is said to be *regular* if $\gamma'(t)$ is not the zero vector for any $t$ in $I$. Points where $\gamma'(t)$ is the zero vector are called *singular points* of $\gamma$.

We consider here smooth and regular plane curves ($n = 2$ above). We shall suppose that the curve $\gamma : I \to \mathbb{R}^2$ is parametrised by arc length and denote the arc length parameter by $s$. Then, $\mathbf{t}(s) = \gamma'(s)$ is a unit tangent vector to $\gamma$. We denote by $\mathbf{n}(s)$ the unit normal vector to $\gamma$ obtained by rotating $\mathbf{t}(s)$ anti-clockwise by an angle of $\pi/2$. It follows from the fact that $\langle \mathbf{t}(s), \mathbf{t}(s) \rangle = 1$ that $\langle \mathbf{t}'(s), \mathbf{t}(s) \rangle = 0$, so

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \tag{1.1}$$

for some smooth function $\kappa(s)$, called *the curvature* of $\gamma$ at $s$.

We have, similarly, $\langle \mathbf{n}'(s), \mathbf{n}(s) \rangle = 0$, so $\mathbf{n}'(s) = \alpha(s)\mathbf{t}(s)$ for some function $\alpha(s)$. Differentiating the identity $\langle \mathbf{t}(s), \mathbf{n}(s) \rangle = 0$ and using (1.1) gives $\alpha(s) = -\kappa(s)$, so that

$$\mathbf{n}'(s) = -\kappa(s)\mathbf{t}(s).$$

We can use (1.1) to deduce that

$$\kappa(s) = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle.$$

When the parameter $t$ of the curve $\gamma$ is not necessarily the arc length parameter, the curvature is given by the formula

$$\kappa(t) = \frac{\det(\gamma'(t), \gamma''(t))}{||\gamma'(t)||^3}.$$

(One can re-parametrise $\gamma$ by arc length and use the chain rule to get the above formula, see for example [Bruce and Giblin (1992)].) If we write $\gamma(t) = (x(t), y(t))$, then

$$\kappa = \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{\frac{3}{2}}},$$

where all the functions are evaluated at $t$.

The curvature function determines completely the curve up to rigid motions (i.e., up to translations and rotations about points in the plane). Indeed,

**Theorem 1.1 (Fundamental Theorem of Plane Curves).**
*Given a smooth function $\kappa(s) : I \to \mathbb{R}$, there is a smooth and regular curve $\gamma : I \to \mathbb{R}^2$ parametrised by arc length $s$ with curvature $\kappa(s)$. The curve $\gamma$ is unique up to rigid motions of $\mathbb{R}^2$.*

An *inflection point* of $\gamma$ is a point where $\kappa(t) = 0$. An inflection point is referred to as an *ordinary inflection* if $\kappa(t) = 0$ but $\kappa'(t) \neq 0$.

A *vertex* of $\gamma$ is a point where $\kappa(t) \neq 0$ and $\kappa'(t) = 0$. A vertex is called an *ordinary vertex* if $\kappa(t) \neq 0$, $\kappa'(t) = 0$ and $\kappa''(t) \neq 0$.

We define the following types of singularities of plane curves.

**Definition 1.1.** (1) A smooth curve $\gamma : I \to \mathbb{R}^2$ has an ordinary cusp singularity at $t_0 \in I$ if $t_0$ is a singular point of $\gamma$ and the vectors $\gamma''(t_0)$ and $\gamma'''(t_0)$ are linearly independent (Figure 1.1, left).

(2) A smooth curve $\gamma : I \to \mathbb{R}^2$ has a $(3, 4)$-singularity at $t_0 \in I$ if $\gamma'(t_0) = \gamma''(t_0) = (0, 0)$ and $\gamma'''(t_0)$ and the fourth derivative vector $\gamma^{(4)}(t_0)$ are linearly independent (Figure 1.1, right).

### 1.1.1 *The evolute of a plane curve*

The *evolute* of a curve $\gamma : I \to \mathbb{R}^2$ is the plane curve $\boldsymbol{\varepsilon}$ parametrised by

$$\boldsymbol{\varepsilon}(t) = \gamma(t) + \frac{1}{\kappa(t)}\mathbf{n}(t), \; t \in I, \tag{1.2}$$

Fig. 1.1 A cusp singularity left and a $(3,4)$-singularity right.

where $\mathbf{n}(t)$ is the unit normal vector obtained by rotating the unit tangent vector $\gamma'(t)/||\gamma'(t)||$ anti-clockwise by $\pi/2$.

The evolute is well defined and is a smooth curve away from the inflection points of $\gamma$. We can use classical differential geometry techniques to study its geometry.

**Proposition 1.1.** *The evolute of a smooth and regular curve $\gamma : I \to \mathbb{R}^2$ is a regular curve except at points corresponding to the vertices of $\gamma$. The evolute has an ordinary cusp singularity at points corresponding to ordinary vertices of $\gamma$.*

***Proof.*** We take $\gamma$ parametrised by arc length $s$. Differentiating (1.2) and dropping the argument $s$, we get

$$\boldsymbol{\varepsilon}' = -\frac{\kappa'}{\kappa^2}\mathbf{n},$$

and this is the zero vector at $s_0 \in I$ if and only if $\kappa'(s_0) = 0$, that is, if and only if $\gamma$ has a vertex at $s_0$.

We obtain by differentiating again

$$\begin{aligned}
\boldsymbol{\varepsilon}'' &= \frac{\kappa'}{\kappa}\mathbf{t} + \left(\frac{2\kappa'^2 - \kappa''\kappa}{\kappa^3}\right)\mathbf{n},\\
\boldsymbol{\varepsilon}''' &= \left(\frac{2\kappa''\kappa - 3\kappa'^2}{\kappa^2}\right)\mathbf{t} + \left(\frac{\kappa'\kappa^4 - \kappa'''\kappa^2 + 6\kappa''\kappa'\kappa - 6\kappa'^3}{\kappa^4}\right)\mathbf{n}.
\end{aligned}$$

At a vertex $s_0$ of $\gamma$, $\kappa'(s_0) = 0$ and the expressions for $\boldsymbol{\varepsilon}''$ and $\boldsymbol{\varepsilon}'''$ at $s_0$ simplify and become

$$\begin{aligned}
\boldsymbol{\varepsilon}''(s_0) &= -\frac{\kappa''(s_0)}{\kappa^2(s_0)}\mathbf{n}(s_0),\\
\boldsymbol{\varepsilon}'''(s_0) &= \frac{2\kappa''(s_0)}{\kappa(s_0)}\mathbf{t}(s_0) - \frac{\kappa'''(s_0)}{\kappa^2(s_0)}\mathbf{n}(s_0).
\end{aligned}$$

The vectors $\boldsymbol{\varepsilon}''(s_0)$ and $\boldsymbol{\varepsilon}'''(s_0)$ are linearly independent if and only if $\kappa''(s_0) \neq 0$. Thus, the evolute of $\gamma$ has an ordinary cusp singularity if and only if the corresponding point on $\gamma$ is an ordinary vertex of $\gamma$. $\square$

An ellipse has four ordinary vertices and Figure 1.2, left, shows the evolute of an ellipse with its four ordinary cusps. The vertices of a curve are points where the curve is most or least curved and these can be detected (approximately) by the naked eye for the ellipse in Figure 1.2, left. It is not possible to do so for the ellipse in Figure 1.2, right, as its principal axes have almost the same length. The ellipse in Figure 1.2, right, looks like a circle but is not a circle as its evolute is not a point. We can find the vertices of the ellipse in Figure 1.2, right, by considering the limiting tangent lines to the evolute at its ordinary cusp. These lines intersect the ellipse at its vertices.

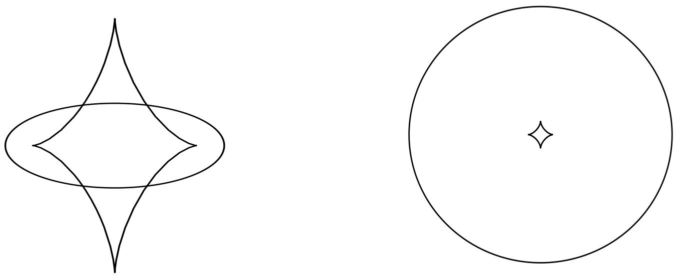

Fig. 1.2 Evolutes of ellipses: the difference between the lengths of the principal axes of the ellipse on the left is noticeable, whereas that of the ellipse on the right is negligible. The ellipse on the right looks like a circle but is not a circle.

### 1.1.2 *Parallels of a plane curve*

A *parallel* (or a wavefront) of a curve $\gamma : I \to \mathbb{R}^2$ is the curve obtained by moving each point on $\gamma$ along its unit normal vector by a fixed distance $d$. When $\gamma$ is parametrised by arc length, a parametrisation of a parallel is given by

$$\rho_d(s) = \gamma(s) + d\mathbf{n}(s),\ s \in I.$$

We have $\rho_d'(s) = (1 - d\kappa(s))\mathbf{t}(s)$, so a parallel is singular at points where $d = 1/\kappa(s)$. This means that the singular points of a parallel are located on the evolute of $\gamma$. As $d$ varies, the singular points of the parallels of $\gamma$

trace the evolute of $\gamma$. Figure 1.3 shows the parallels of an ellipse with their singular points tracing the evolute of the ellipse.

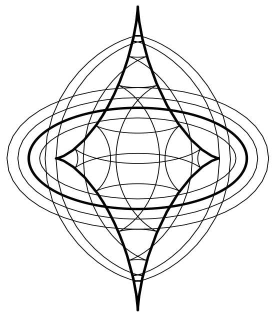

Fig. 1.3 The parallels of an ellipse. The ellipse and its evolute are drawn in thick.

**Proposition 1.2.** *The s of a smooth and regular curve $\gamma$ have an ordinary cusp singularity at regular points on the evolute. The parallel through an ordinary singularity of the evolute has a $(3,4)$-singularity at that point.*

***Proof.*** We have the following successive derivatives of the parametrisation of a parallel:

$$\begin{aligned}
\rho_d' &= (1-d\kappa)\mathbf{t},\\
\rho_d'' &= -d\kappa'\mathbf{t} + \kappa(1-d\kappa)\mathbf{n},\\
\rho_d''' &= -(d\kappa'' + \kappa^2(1-d\kappa))\mathbf{t} + \kappa'(1-3d\kappa)\mathbf{n},\\
\rho_d^{(4)} &= -(d\kappa''' + 3\kappa'\kappa(1-2d\kappa))\mathbf{t}\\
&\quad +(-\kappa(d\kappa'' + \kappa^2(1-d\kappa)) + \kappa''(1-3d\kappa) - 3d\kappa'^2)\mathbf{n}.
\end{aligned}$$

At a singularity $s_0$ of the parallel $d = 1/\kappa(s_0)$, so

$$\begin{aligned}
\rho_d''(s_0) &= -\frac{\kappa'(s_0)}{\kappa(s_0)}\mathbf{t}(s_0),\\
\rho_d'''(s_0) &= -\frac{\kappa''(s_0)}{\kappa(s_0)}\mathbf{t}(s_0) - 2\kappa'(s_0)\mathbf{n}(s_0).
\end{aligned}$$

The vectors $\rho_d''(s_0)$ and $\rho_d'''(s_0)$ are linearly independent if and only if $\kappa'(s_0) \neq 0$, equivalently, if and only if $\gamma(s_0)$ is not a vertex of $\gamma$. If this is the case, the parallel $\rho_d$ with $d = 1/\kappa(s_0)$ has an ordinary cusp singularity at $s = s_0$ (see Definition 1.1).

Suppose that $d = 1/\kappa(s_0)$ and $\kappa'(s_0) = 0$. Then $\rho_d'(s_0) = \rho_d''(s_0) = 0$ and

$$\begin{aligned}\rho_d'''(s_0) &= -\frac{\kappa''(s_0)}{\kappa(s_0)}\mathbf{t}(s_0),\\ \rho_d^{(4)}(s_0) &= -\frac{\kappa'''(s_0)}{\kappa(s_0)}\mathbf{t}(s_0) - 3\kappa''(s_0)\mathbf{n}(s_0).\end{aligned}$$

The vectors $\rho_d'''(s_0)$ and $\rho_d^{(4)}(s_0)$ are linearly independent if and only if $\kappa''(s_0) \neq 0$, equivalently, if and only if $\gamma(s_0)$ is an ordinary vertex of $\gamma$. If this is the case, the parallel $\rho_d$ with $d = 1/\kappa(s_0)$ has a $(3,4)$-singularity at $s_0$ (see Definition 1.1). □

### 1.1.3 *The evolute from the singularity theory viewpoint*

In sections 1.1.1 and 1.1.2 we obtained geometric information about the evolute and parallels of a plane curve by direct computation of the successive derivatives of a parametrisation of the curve. This method has several limitations. For instance, it does not explain which singularities could appear in the evolute and parallels and how these bifurcate as the original curve is deformed. It also misses to capture the deep concepts involved. We outline these concepts in this section.

We consider the *contact* (see Chapter 4) of a smooth and regular plane curve $\gamma : I \to \mathbb{R}^2$ with circles. A circle of centre $\boldsymbol{a}$ and radius $r$ is the level set $D_a(p) = r^2$ of the *distance squared function* $D_a : \mathbb{R}^2 \to \mathbb{R}$, given by

$$D_a(p) = ||p - a||^2 = \langle p - a, p - a\rangle.$$

The contact of $\gamma$ with the level sets of $D_a$ can be measured by the vanishing of successive derivatives of the function

$$g(s) = D_a(\gamma(s)) = \langle \gamma(s) - a, \gamma(s) - a\rangle.$$

A point $\gamma(s_0)$ is on a circle $C$ of centre $a$ and radius $r$ if and only if $g(s_0) = r^2$. The curve $\gamma$ and the circle $C$ have an ordinary tangency at $\gamma(s_0)$ if and only if $g(s_0) = r^2$, $g'(s_0) = 0$ and $g''(s_0) \neq 0$. Higher orders of tangency between $\gamma$ and $C$ are captured by the vanishing of the successive derivatives of $g$ at $s_0$.

**Definition 1.2.** We say that $\gamma$ and $C$ have $k+1$-*point contact* at $s_0$ if $g^{(i)}(s_0) = 0$ for $i = 1, \ldots, k$ but $g^{(k+1)}(s_0) \neq 0$. Then $s_0$ is said to be a

Table 1.1 Geometric conditions for the singularities of $g$.

| $g$ | Conditions | Geometric interpretation |
|---|---|---|
| $A_1$ | $a = \gamma(s_0) + \lambda \mathbf{n}(s_0)$, $\lambda \neq \frac{1}{\kappa(s_0)}$ | The centre of the circle $C$ lies on the normal line to $\gamma$ at $s_0$ but is not on the evolute of $\gamma$. |
| $A_2$ | $a = \gamma(s_0) + \frac{1}{\kappa(s_0)}\mathbf{n}(s_0)$, $\kappa'(s_0) \neq 0$ | The centre of the circle $C$ lies on the evolute of $\gamma$ but $s_0$ is not a vertex of $\gamma$. |
| $A_3$ | $a = \gamma(s_0) + \frac{1}{\kappa(s_0)}\mathbf{n}(s_0)$, $\kappa'(s_0) = 0$, $\kappa''(s_0) \neq 0$ | The centre of the circle $C$ lies on the evolute of $\gamma$ and $s_0$ is an ordinary vertex of $\gamma$. |

singularity of $g$ of type $A_k$. We say that $\gamma$ and the circle $C$ have $\geq$ *k-point contact* at $s_0$ if $g^{(i)}(s_0) = 0$ for $i = 1, \ldots, k$ and call $s_0$ a singularity of $g$ of type $A_{\geq k}$.

Suppose that $g(s_0) = r^2$. We can recover geometric information about the curve $\gamma$ at $s_0$ from the singularity type of the function $g$ at $s_0$.

**Proposition 1.3.** *Let $\gamma : I \to \mathbb{R}^2$ be a smooth and regular plane curve and let $C$ be a circle of centre $a$ and radius $r$. Suppose that $g(s_0) = r^2$ for some $s_0 \in I$. Then $g$ has a singularity of type $A_1, A_2$ or $A_3$ at $s_0$ if and only if the geometric conditions in Table 1.1 are satisfied.*

***Proof.*** We take $\gamma$ parametrised by arc length. Then the result follows by observing that

$$\begin{aligned}
\tfrac{1}{2}g' &= \langle \mathbf{t}, \gamma - a\rangle, \\
\tfrac{1}{2}g'' &= \kappa\langle \mathbf{n}, \gamma - a\rangle + 1, \\
\tfrac{1}{2}g''' &= \kappa'\langle \mathbf{n}, \gamma - a\rangle - \kappa^2\langle \mathbf{t}, \gamma - a\rangle, \\
\tfrac{1}{2}g^{(4)} &= (\kappa'' - \kappa^3)\langle \mathbf{n}, \gamma - a\rangle - 3\kappa\kappa'\langle \mathbf{t}, \gamma - a\rangle - \kappa^2,
\end{aligned}$$

where all the functions are evaluated at $s$. □

It is possible to carry on and identify geometrically the singularities of $g$ of type $A_k$, with $k > 3$. However, in general, or to be more precise for *generic* curves, the function $g$ has only singularities of type $A_1$, $A_2$ or $A_3$. (The concept of genericity is dealt with in Chapter 4. Intuitively, a property of an object is generic if it persists when the object is deformed.)

We consider the functions $D_a$, $a \in \mathbb{R}^2$, all together as members of the *family of distance squared functions* $D : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$, given by

$$D(p, a) = D_a(p).$$

The restriction of $D$ to a plane curve $\gamma$ is the family $D : I \times \mathbb{R}^2 \to \mathbb{R}$, given by

$$D(s,a) = \langle \gamma(s) - a, \gamma(s) - a \rangle.$$

The *catastrophe set* of the family $D$ is defined to be the set

$$C_D = \left\{ (s,a) \in I \times \mathbb{R}^2 \, : \, \frac{\partial D}{\partial s}(s,a) = 0 \right\},$$

and the (local) *bifurcation set* of $D$ is defined as the set

$$\begin{aligned} B_D &= \left\{ a \in \mathbb{R}^2 \, : \, \exists (s,a) \in C_D \text{ such that } \tfrac{\partial^2 D}{\partial s^2}(s,a) = 0 \right\} \\ &= \left\{ a \in \mathbb{R}^2 \, : \, D_a \text{ has an } A_{\geq 2} - \text{singularity at some } s \in I \right\}. \end{aligned}$$

**Proposition 1.4.** (1) *The local bifurcation set of the family of distance squared functions on $\gamma$ is the evolute of $\gamma$.*

(2) *The catastrophe set $C_D$ is a regular surface in $I \times \mathbb{R}^2$. The set of critical values of the catastrophe map $\pi_{C_D} : C_D \to \mathbb{R}^2$, with $\pi_{C_D}(s,a) = a$, is the local bifurcation set of the family of distance squared functions.*

***Proof.*** The proof of (1) follows from the definition of the bifurcation set and from Proposition 1.3. As for (2), we prove in Chapter 5 a more general result that shows that $C_D$ is a regular surface. □

We can now outline the underlying singularity theory concepts involved in the study of the evolute. These are developed in subsequent chapters.

1. The cotangent bundle $T^*\mathbb{R}^2$, with the canonical projection $\pi : T^*\mathbb{R}^2 \to \mathbb{R}^2$ to the base space, has the canonical symplectic structure $\omega = \sum_{i=1}^2 dq_i \wedge dp_i$.
2. There is a Lagrangian immersion $L(D) : C_D \to T^*\mathbb{R}^2$, that is, $L(D)$ is an immersion and $L(D)(C_D)$ is a Lagrangian surface in $T^*\mathbb{R}^2$.
3. The following diagramme commutes

$$\begin{tikzcd} & T^*\mathbb{R}^2 \arrow[d, "\pi"] \\ C_D \arrow[ur, "L(D)"] \arrow[r, "\pi_{C_D}"'] & \mathbb{R}^2 \end{tikzcd}$$

so the catastrophe map $\pi_{C_D} = \pi \circ L(D)$ is a Lagrangian map.
4. It follows from Proposition 1.4 that the evolute is the set of critical values of the Lagrangian map $\pi_{C_D}$, i.e., it is a *caustic*. As a consequence, it has only Lagrangian singularities.